Random graphs, finite extension constructions, and complexity

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October 4, 2015

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Homogeneous structures

- A countable (relational) structure $\mathcal{M}$ is *homogeneous* if every isomorphism between finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.
- **Fraissé**: Any homogeneous structure arises as a *amalgamation process* of finite structures over the same language (Fraissé limits).
- Examples:
  - $(\mathbb{Q}, <)$,
  - the Rado (random) graph
  - the universal K_n-free graphs, $n \geq 3$ (Henson)

Randomized constructions

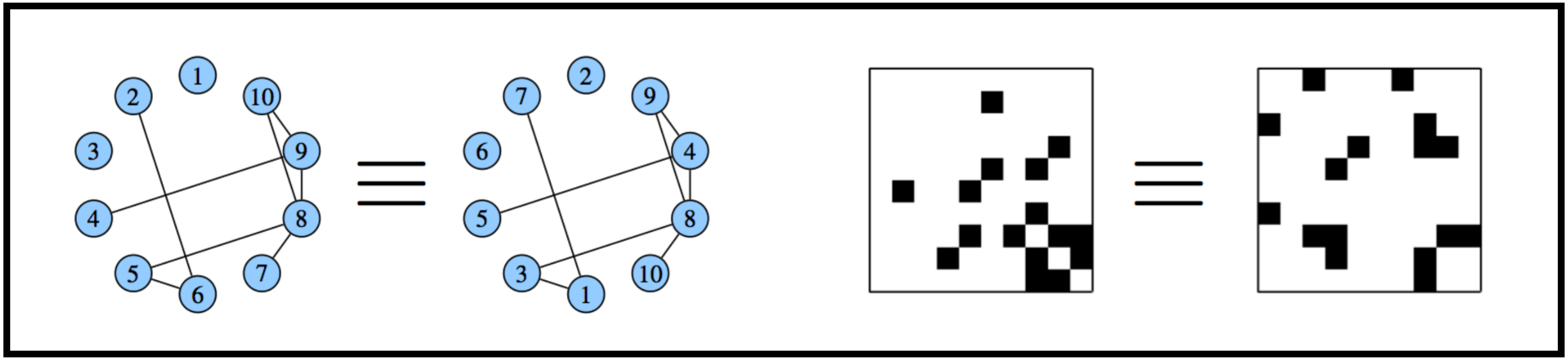
- Many homogeneous structures can obtained (almost surely) by adding new points according to a randomized process.
  - $(\mathbb{Q}, <)$: add the n-th point between (or at the ends) of any existing point with uniform probability $1/n$.
  - Rado graph: add the n-th vertex and connect to every previous vertex with probability p (uniformly and independently).
  - Vershik (2004): Urysohn space,
Droste and Kuske (2003): universal poset
  - Henson graph: ??? (until 2008)

Constructions "from below"

- A naive approach to "randomize" the construction of the Henson graph would be as follows:
  - In the n-th step of the construction, pick a one-vertex extension uniformly among all possible extensions that preserve K_n-freeness.
- However: **Erdös, Kleitman, and Rothschild** (1976) showed that this asymptotically almost surely yields a bipartite graph (in fact, the *universal* countable bipartite graph).
  - The Henson graph(s), in contrast, has to contain C_5 and hence cannot be bipartite.

Symmetric constructions

- On the other hand, one could (degenerately) ensure that every triangle-free subgraph appears, and indeed witness all extension axioms, by deterministically building the Henson graph.
- But this violates symmetry: we would like the joint distribution of any distinct k-tuple to be the same as any other (i.e., exchangeability).



from Lloyd–Orbanz–Ghahramani–Roy (2012), Random function priors for exchangeable arrays from Lloyd–Orbanz–Ghahramani–Roy (2012), Random function priors for exchangeable arrays

Symmetric constructions

- **Is there an exchangeable construction of the Henson graph?**
- Equivalently, is there a probability measure on graphs with vertex set ω that is concentrated on the isomorphism class of the Henson graph, and is invariant under the logic action of the symmetric group S_∞ on the underlying set ω of vertices?

Constructions "from above"

- **Petrov and Vershik** (2010) showed how to construct universal K_n-free graphs probabilistically by *sampling them from a continuous graph*.
- Indeed every exchangeable structure in a countable language must arise in essentially this way, as shown by Aldous (1981) and Hoover (1979).
- These continuous graphs, known as **graphons**, have been studied extensively over the past decade.
  - See, for example the recent book by Lovasz, *Large networks and graph limits* (2012).

Graphons

- One basic motivation behind graphons is to capture the asymtotic behavior of growing sequences of dense graphs, e.g. with respect to subgraph densities.
- While the Rado graph can be seen as the limit object of a sequence (G_n) of finite random graphs, it does not distinguish between the distributions with which the edges are produced.
- For any $0 < p < 1$, $\mathbb{G}(n, p)$ "converges" almost surely to (an isomorphic copy of) the Rado graph.
    - However, if $p_1 \ll p_2$, $\mathbb{G}(n, p_1)$ will exhibit subgraph densities very different from $\mathbb{G}(n, p_2)$

Convergence

- Let (G_n) be a graph sequence with $|V(G_n)| \to \infty$.
- We say (G_n) **converges** if

for every finite graph F, the relative number $t_i(F, G_n)$ of embeddings of F into G_n converges.

Graphons

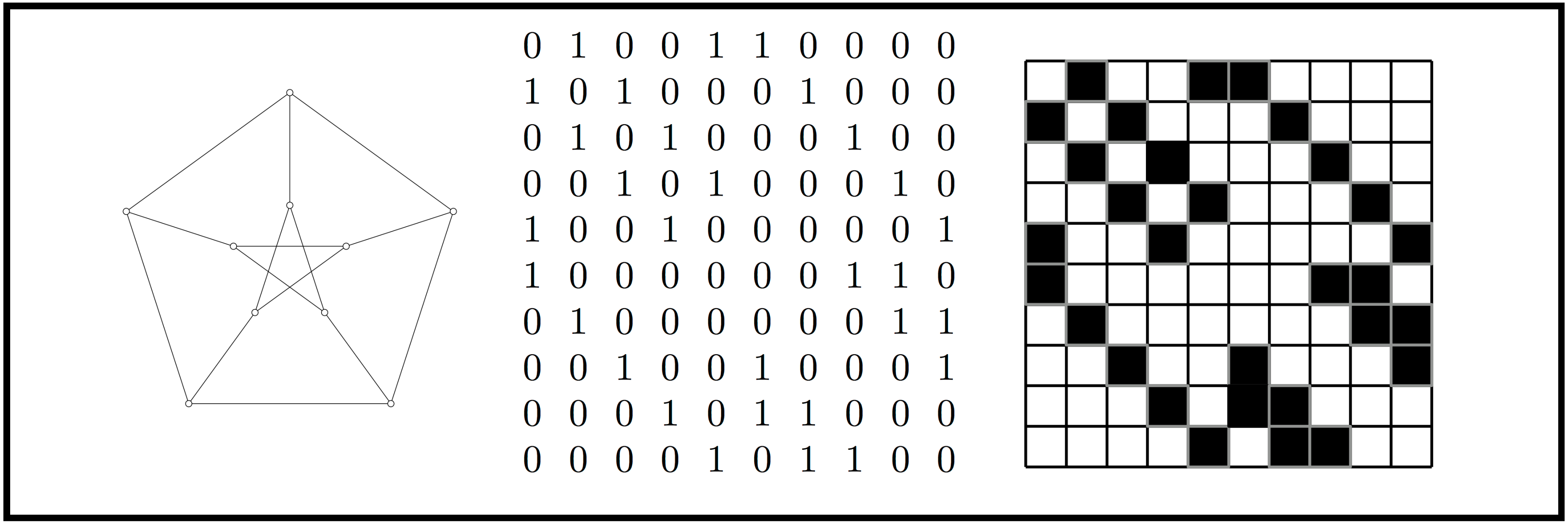
- $W : [0,1]^2 \to [0,1]$ measurable, and for all x, y,

$$W(x,x) = 0 \text{ and } W(x,y) = W(y,x).$$

- Think: $W(x,y)$ is the probability there is an edge between x and y.
- Subgraph densities:
  - edges: $\int W(x,y)\,dx\,dy$
  - triangles: $\int W(x,y)W(y,z)W(z,x)\,dx\,dy\,dz$
  - this can be generalized to define $t_i(F,W)$.

Graphons and graph limits

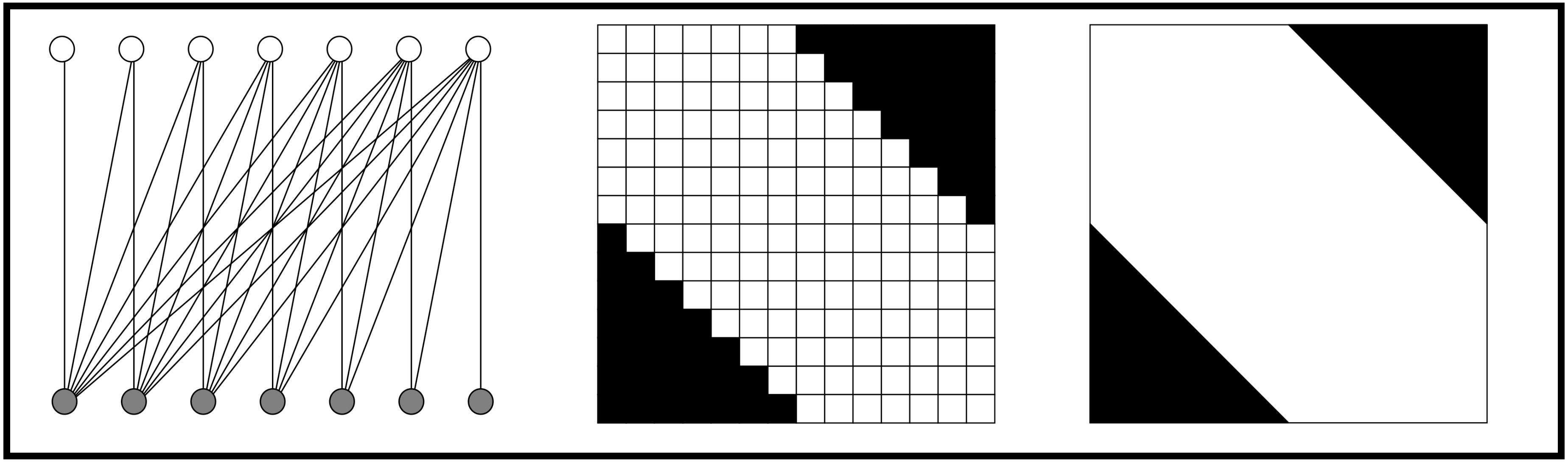
Basic idea: "**pixel pictures**"



from Lovasz (2012), Large networks and graph limits

Graphons and graph limits

Convergence of pixel pictures



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Graphons and graph limits

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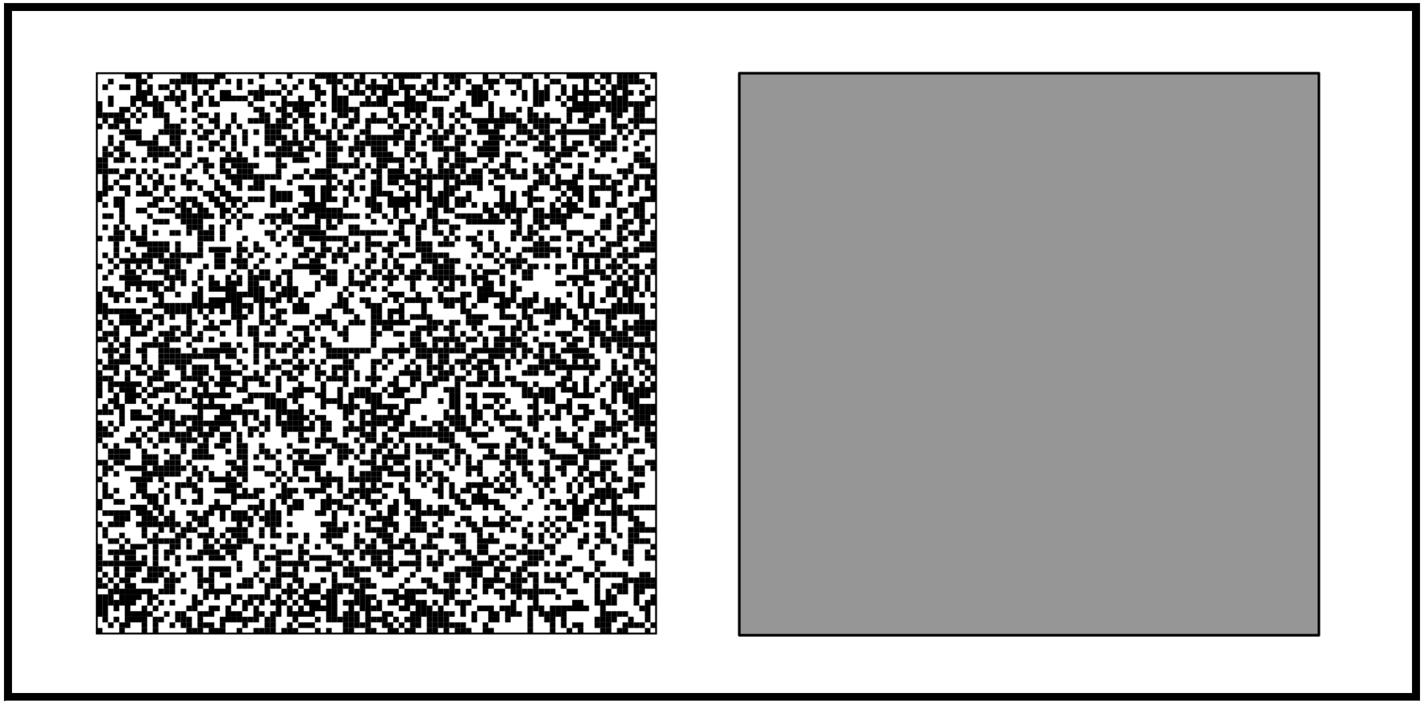

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The limit graphon

THM: For every convergent graph sequence (G_n) there exists (up to weak isomorphim) exactly one graphon W such that for all finite F:

$$t_i(F, G_n) \longrightarrow t_i(F, W).$$

Sampling from graphons

- We can obtain a finite graph $\mathbb{G}(n, W)$ from W by (independently) sampling n points $x_1, \ldots, x_n$ from $[0, 1]$ and filling edges according to probabilities $W(x_i, x_j)$.
  - almost surely, we get a sequence with $\mathbb{G}(n, W) \to W$.
- If we sample ω-many points from $W(x, y) \equiv 1/2$, we almost surely get the random graph.

The Petrov-Vershik graphon

- **Petrov and Vershik** (2010) constructed, for each $n \geq 3$, a graphon W such that we almost surely sample a Henson graph for n.
  - The graphons are (necessarily) {0,1}-valued.
  - Such graphons are called **random-free**.
  - The constructions resembles a finite extension construction with simple geometric forms, where each step satisfies a new type requiring attention.
  - The method can also be used to construct random-free graphons from which we sample the Rado graph.

Invariant measures

- The Petrov-Vershik graphon also yields a measure on the set of countable infinite graphs concentrating on the set of universal, homogeneous K_n-free graphs.
- This measure will be invariant under the "logic action", the natural action of S_∞ on the space of countable (relational) structures with universe $\mathbb{N}$.
- This method was generalized by *Ackerman, Freer, and Patel* (2014) to other homogeneous structures.
- It can be used to define algorithmic randomness for such structures (as suggested by Nies and Fouché).

Universal graphons

- A random-free graphon is *countably universal* if for every set of distinct points from [0, 1], $x_1, x_2, \dots, x_n, y_1, \dots, y_m$, the intersection
$$\bigcap_{i,j} E_{x_i} \cap E_{y_j}^C$$
has non-empty interior.
  - Here $E_x = \{y : W(x, y) = 1\}$ is the neighborhood of x.

- For *countably K_n-free universal* graphs, we require this to hold only for such tuples where the induced subgraph by the x_i has no induced K_{n-1}-subgraph,
  - also require that no n-tuples induce a K_n.

The topology of graphons

- Neighborhood distance:

  $$r_W(x, y) = \| W(x, .) - W(y, .) \|_1 = \int |W(x, z) - W(y, z)| \, dz$$

  and mod out by $r_W(x, y) = 0$.

- Example: $W(x, y) \equiv p$ is a singleton space.

- **THM: (Freer & R.)** (informal) If W is a random-free universal graphon obtained via a "tame" extension method, then W is not compact in the r_W topology.

"Tame" extensions

- **DEF:** A random-free graphon W has *continuous realization of extensions* if there exists a function
$$f : (x_1, \dots, x_n), (y_1, \dots, y_m) \mapsto (l, r)$$
that is continuous a.e. such that for all $\vec{x}, \vec{y}$,
$$[l, r] \subseteq \bigcap_{i,j} E_{x_i} \cap E_{y_j}^C.$$
- Here $E_x = \{y\colon W(x, y) = 1\}$ is the neighborhood of x.
- The Petrov-Vershik graphons have uniformly continuous realization of extensions.

Non-compactness

THM: *If a countably (K_n-free) universal graphon has uniformly continuous realization of extensions, then it is not compact in the r_W-topology.*

Compactness of graphons

- This contrasts the following result due to Lovasz and Szegedy.
- **THM:** If a pure graphon (J, W) misses some signed bipartite graph F, then

  (i) (J, r_W) is compact, and
  (ii) has Minkowski dimension at most $10v(F)$.

Complexity of universal graphons

construction:	**fully random**	**tame deterministic**	**general deterministic**
complexity of graphon	**low** (singleton)	**high** (not compact, infinite Minkowski dimension)	**?**