

Algorithmic Randomness

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Outline of the Lectures

Lecture 1: Martin-Löf tests and martingales.

Lecture 2: Kolmogorov complexity.

Lecture 3: The computational power of randomness.

Lecture 4: Randomness for non-uniform distributions.

Lecture 5: Randomness in Fractal Geometry and Dynamical Systems.

Lecture 6: Lowness and Triviality

Some Basics

Books

- Downey and Hirschfeldt, *Algorithmic Randomness and Complexity* (Springer)
- Li and Vitanyi, *An Introduction to Kolmogorov Complexity and its Applications* (Springer)
- Nies, *Randomness and Computability* (Oxford UP)

Prerequisites

- *Basic computability theory* – Turing machines, computable sets, halting problem, Turing jump, r.e. sets
- *Basic mathematical logic*, some set theory later on
- *Basic measure theory* (helpful)

Initial Clarifications

Logic

- We are particularly concerned with **definability**, i.e. the relation between a language and mathematical objects we can describe.
- Our language will mostly be that of (second order) **arithmetic**, $+$, \cdot , $=$, \in .
- By the well-known correspondence between definability in arithmetic and (relative) computability, our study of randomness is also referred to as **algorithmic randomness**.

Initial Clarifications

Randomness

Our goal: define what an **individual random object is**.

Think of an outcome of an infinite sequence of coin tosses.

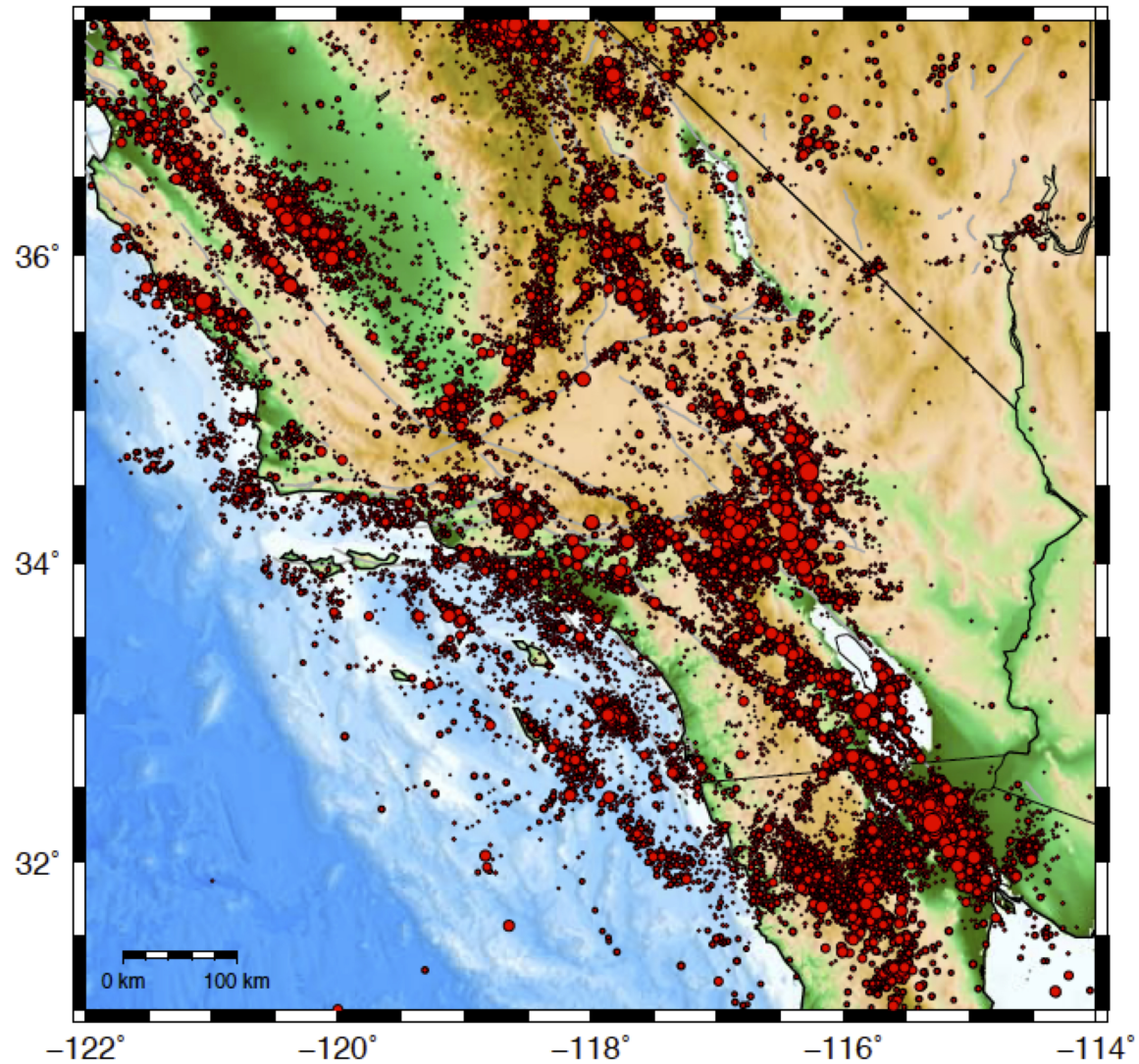
Three Paradigms

- **Typicalness**: A random object is the typical outcome of a random variable.
- **Unpredictability**: A random object should be impossible to predict.
- **Incompressibility**: A random object should not have a shorter description than itself.

In the following, we want to give a sound meaning to these paradigms.

Initial Clarifications

Do earthquakes occur randomly?



Hauksson-Shearer-Yang catalog of southern CA earthquakes 1981-2011

Initial Clarifications

If we do not **restrict the methods** allowed for betting, compressing, etc., we easily end up in paradox:

- A typical outcome should satisfy all **probabilistic laws**, such as the Law of Large Numbers.
- A probabilistic law is essentially a set of measure 1.
- However, the intersection of all sets of measure 1 is empty!

Initial Clarifications

A different example:

- A random sequence X of coin tosses should be unpredictable – there should be no prediction function that, given as input a finite sequence $X(1) \dots X(n)$ of outcomes so far, it predicts the next bit of X correctly.
- However, there clearly is such a prediction function – one that, on any input of length n , simply outputs the $n + 1$ -st bit of X .

Initial Clarifications

Remedy:

Admit only

definable laws, betting strategies, compression methods.