Effective Geometric Measure Theory

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Geometric Measure Theory

- Study the geometric properties of non-negligible closed subsets of perfect Polish spaces.
- Non-negligible: with respect to a translation-invariant non-atomic measure (Hausdorff measure).
- Problem: In general, these measures are not σ-finite, there is no integration theory, etc.
- Basic tool: replace the Hausdorff measure by a probability measure "sufficiently close" to it.
- Major goal: understand local structure of fractals - density, rectifiability, etc.

Effective Geometric Measure Theory

- Local structure: study typical points in a set.
- Use both measure theory and recursion theory / effective descriptive theory
- This amounts to analyzing infinite paths through trees.

Types of Measures

Probability measures

Based on a premeasure ρ which satisfies

- $\rho(\emptyset) = 1$ and
- $\rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)$.

For probability measures it holds that $\mu_\rho(N_\sigma) = \rho(\sigma)$.

Examples

- Lebesgue measure λ: $\rho(\sigma) = 2^{-|\sigma|}$.

If $\mu(\{X\}) = 0$ for all X, then μ is called continuous.

Types of measures

Hausdorff premeasures

Any premeasure $\rho : 2^{<\omega} \to \mathbb{R}^{\geqslant 0}$ satisfying

- If $|\sigma| = |\tau|$, then $\rho(\sigma) = \rho(\tau)$.
- $\rho(n)$ is non-increasing.
- $\rho(n) \to 0$ as $n \to \infty$.
- For example: $\rho(\sigma) = 2^{-|\sigma|s}$, $s \geqslant 0$.

Geometrical premeasures

There exist real numbers p, q

- $1/2 \leqslant p < 1$ and $1 \leqslant q < 2$;
- $\rho(\sigma^\frown i) \leqslant p\rho(\sigma)$;
- $q\rho(\sigma) \leqslant \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)$.

Ranked points

Small trees - countable Π^0_1 classes

- It is not hard to see that such trees can neither support a probability nor a Hausdorff measure.
- It is known that members of such trees are hyperarithmetical, but instances occur at all levels of the hyperarithmetical hierarchy. [Kreisel]
- No member of a countable Π^0_1 class is random for a continuous probability measure. [Kjos-Hanssen and Montalban]
- Let

$$\mathrm{NCR}_1 = \{X\text{: X not random for any cont. prob. measure }\}$$

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Uncountable Sets

Question

Under what circumstances does a set support a continuous probability or Hausdorff measure?

- Probability measures: If the set is regular (Borel), then uncountability suffices – perfect subset property.
- Hausdorff measures: It is possible to “translate” between Hausdorff and certain kinds of probability measures.

Non-ranked points

Question

In the effective case – is being an element of a countable effectively closed set also necessary for not being continuously random?

- There was some evidence for this, since NCR_1 is completely contained in Δ^1_1.

Theorem [Reimann and Slaman]

There exists an $X \in \mathrm{NCR}_1$ that is not a member of any countable Π^0_1 class.

Non-ranked instances in NCR_1

Lemma

If a recursive tree T does not contain a recursive path, then no member of $[T]$ can be an element of a countable Π^0_1 class.

- Intersect a countable Π^0_1 class with T.
- If the resulting tree S had an infinite path, $[S]$ would be a countable Π^0_1 class without a recursive path.
- But this is impossible.

Non-ranked instances in NCR_1

Given a tree T, let

$$T_\infty = \{\sigma \in T : \sigma \text{ has inf. many ext. in } T\},$$

the infinite part of T.

Lemma

There exists a recursive tree T such that T has no recursive path and for all $\sigma \in T_\infty$, if there exist n branches along σ, then $0' \upharpoonright_n$ is settled by stage $|\sigma|$.

- Essentially a priority argument, diagonalizing against recursive paths and "thinning out" T_∞ if the current approximation to $0'$ changes.

Non-ranked instances in NCR_1

T is recursive, hence $[T] = [T_\infty]$ contains an element in Δ^0_2, say X.

Assume X is μ-random for some continuous μ.

- T is recursive and contains a μ-random path, hence $\mu[T] > 0$.
- Recursively in μ, we can compute $h : \mathbb{N} \to \mathbb{N}$ such that for each n, some element in $[T]$ must have n-many branchings in T_∞ by level $h(n)$.
- But this implies that μ computes $0'$, hence μ computes X, a contradiction!

Non-ranked points

Observation

- The tree T used in the previous proof splits very slowly.
- In terms of Hausdorff measure this means that $[T]$ is $\mathcal{H}^h$-null for any recursive Hausdorff premeasure h.

Binns suggested that it might be precisely such trees that capture being NCR_1.

- There is evidence for this based on a correspondance between Hausdorff and probability measures.

Hausdorff and Probability Measures

Support of a probability measure

$\text{supp}(\mu)$ is the smallest closed set F such that $\mu(2^\omega \setminus F) = 0$.

$A \subseteq 2^\omega$ supports a measure μ if $\text{supp}(\mu) \subseteq A$.

Mass Distribution Principle

If A supports a probability measure μ such that for some constant $c > 0$,

$$(\forall \sigma)\ \mu(\sigma) \leqslant c2^{-|\sigma| s},$$

then $\mathcal{H}^s(A) > \mu(A)/c$.

Hausdorff and Probability Measures

A fundamental result due to Frostman (1935) asserts that the converse holds, too.

Frostman's Lemma

If A is closed and $\mathcal{H}^s(A) > 0$, then there exists a probability measure μ such that $\operatorname{supp}(\mu) \subseteq A$ and for some $c > 0$,

$$(\forall\sigma)\ \mu(\sigma) \leqslant c2^{-|\sigma|s}.$$

(Call such a measure s-bounded.)

We will prove a pointwise version of Frostman's Lemma.

Randomness and Complexity

- An order is a nondecreasing, unbounded function $h : \mathbb{N} \to \mathbb{N}$. h is called convex if for all n, $h(n+1) \leqslant h(n) + 1$.
- A real is called complex if for a computable order h

$$(\forall n)\ K(x\restriction_n) \geqslant h(n),$$

  where K denotes prefix-free Kolmogorov complexity.
- If X is complex via h, then we call X h-complex. X is h-complex if and only if it is $\mathcal{H}^{2^{-h}}$-random.

Variants of Complexity

- Replace K by another type of Kolmogorov complexity.
- A (continuous) semimeasure is a function $\eta : 2^{<\omega} \to [0,1]$ such that
$$(\forall\sigma)\, \eta(\sigma) \geqslant \eta(\sigma^\frown 0) + \eta(\sigma^\frown 1).$$
- There exists a maximal enumerable semimeasure $\overline{M}$ that dominates (up to a multiplicative constant) any other enumerable semimeasure (Levin).
- The a priori complexity of a string σ is defined as $-\log \overline{M}(\sigma)$.
- Given a computable order h, we say a real $X \in 2^\omega$ is strongly h-complex if
$$(\forall n)\, [-\log \overline{M}(x\restriction_n) \geqslant h(n)],$$
(Note that up to an additive constant, $-\log \overline{M} \leqslant K$.)

A pointwise version of Frostman's Lemma

Given an order h, we say X is h-capacitable if there exists an h-bounded probability measure μ such that X is μ-random.

Effective Capacitability Theorem

Suppose $X \in 2^\omega$ is strongly h-complex, where h is a computable, convex order function. Then X is h-capacitable.

Applications of Effective Capacitability

- A new proof of Frostman's Lemma.
- A new characterization of effective dimension.
- Comparison of randomness notions.

A New Proof of Frostman's Lemma

The new proof is of a profoundly effective nature.

- Kucera-Gacs Theorem (does not have a classical counterpart)
- Compactness is used in the form of a basis result for Π^0_1 classes.
- The problem of assigning non-trivial measure to A is solved by making an element of A random.

Kjos-Hanssen observed that strong randomness is the precise effective level for which a pointwise Frostman Lemma holds.

Theorem

If X is not strongly $\mathcal{H}^h$-random then X is not effectively h-capacitable.

A New Characterization of Effective Dimension

We also obtain a new characterization of effective dimension.

Theorem

For any real $X \in 2^\omega$,

$$\dim^1_H x = \sup\{s \in \mathbb{Q} : x \text{ is } h\text{-capacitable for } h(n) = sn\}.$$

Comparison of Randomness Notions

Particularly with regard to effective dimension notions, several other test concepts have been suggested.

The standard structure of such tests is as follows: A notion of randomness $\mathcal{R}$ is a uniform mapping

$$\mathcal{R}: \rho \quad \mapsto \quad W \quad \mapsto \quad \bigcap_n W_n,$$

where $W \subset \mathbb{N} \times 2^{<\omega}$ is c.e. in (a representation of) ρ, and $\bigcap_n W_n$ is a μ_ρ-nullset that is $\Pi^0_2(\rho)$.

Comparison of Randomness Notions

Examples:

- Martin-Löf tests: $\rho(W_n) \leqslant 2^{-n}$
- Solovay tests: $W_1 \supseteq W_2 \supseteq \ldots$, W_n contains only strings of length $\geqslant n$ and $\rho(W_n) \leqslant 1$.
- Strong tests (Calude et al): If $V \subseteq W_n$ is prefix-free, then $\rho(V) \leqslant 2^{-n}$.
- Vehement tests (Kjos-Hanssen): For each n exists V_n such that $N(V_n) \supseteq N(W_n)$ and $\rho(V_n) \leqslant 2^{-n}$.

Comparison of Randomness Notions

Say for two notions of randomness that $\mathcal{R}_0 \succeq_\rho \mathcal{R}_1$ if every ρ-random for $\mathcal{R}_0$ is also ρ-random for $\mathcal{R}_1$.

The following relations are known:

- $\mathcal{R}_{ML} \preceq_{\mathcal{P} \cup \mathcal{H}} \mathcal{R}_S$.
- For all computable geometrical premeasures ρ for which (G3) holds for some $q > 1$, $\mathcal{R}_{ML} \not\preceq_\rho \mathcal{R}_S$ (Reimann and Stephan).
- $\mathcal{R}_S \preceq_{\mathcal{G}} \mathcal{R}_{\text{str}}$ (R-S).
- For any computable, length-invariant, geometrical premeasure ρ, $\mathcal{R}_S \not\preceq_\rho \mathcal{R}_{\text{str}}$ (R-S).
- $\mathcal{R}_{\text{str}} \preceq_{\mathcal{P} \cup \mathcal{H}} \mathcal{R}_\nu$ (every open covering in 2^ω has a prefix-free subcovering).

($\mathcal{P}$ denotes the set of all probability measures, $\mathcal{H}$ the set of convex Hausdorff premeasures, and $\mathcal{G}$ the set of all geometrical premeasures.)

Comparison of Randomness Notions

We can use the effective capacitability theorem to show that strong randomness $\mathcal{R}_{\text{str}}$ is the strongest possible randomness notion among a family of "Martin-Löf like" randomness concepts (satisfying certain consistency requirements).

Let $\mathcal{H}^*$ denote the family of all computable convex Hausdorff premeasures.

Theorem

Suppose $\mathcal{R}$ is a randomness notion such that $\mathcal{R}_{\text{str}} \preceq_{\mathcal{H}^*} \mathcal{R}$ and $\mathcal{R}_{\text{str}} \equiv_{\mathcal{P}} \mathcal{R}$. Then $\mathcal{R}_{\text{str}} \equiv_{\mathcal{H}^*} \mathcal{R}$.

Corollary

$\mathcal{R}_{\text{str}} \equiv_{\mathcal{P} \cup \mathcal{H}^*} \mathcal{R}_{\mathcal{V}}$, i.e. for probability and computable Hausdorff measures, strong and vehement randomness coincide.

Randomness and the Size of Trees

Question

Is not being random for a continuous measures related to being a member of some “small” effective tree?

The previous results suggest that this might be the case.

- To answer the question, we will analyze the reals in $\mathrm{NCR}_1 \cap \Delta^0_2$.

$\mathrm{NCR}_1 \cap \Delta^0_2$

We will compare two functions.

- The settling function for $X \in \Delta^0_2$.
  Let X_0 be a recursive approximation $X = \lim_{s\to\infty} X_0(s)$.
  Define $c : \mathbb{N} \to \mathbb{N}$ as

  $$c_X(n) = \text{the least } s \text{ such that for all } t > s,\ X_0(n,t) = X(n).$$

  X can be computed from any function g which dominates c_X pointwise.

- The granularity function of a continuous measure μ. Let $g_\mu : \mathbb{N} \to \mathbb{N}$ be given as

  $$g_\mu(n) = \text{the least } l \text{ such that for all } \sigma \text{ of length } l,\ \mu(\sigma) < 1/2^n.$$

$\mathrm{NCR}_1 \cap \Delta^0_2$

Given these two functions, we can analyze whether $X \in \mathrm{NCR}_1$.

- If g_μ dominates c_X pointwise, then X is recursive in μ and hence not random relative to μ.
- An argument along this line shows, if g_μ is not eventually dominated by c_X, then X can be approximated in measure and is not random relative to μ.

Theorem [Reimann and Slaman]

For each Δ^0_2 set X, there is an arithmetically defined sequence of compact sets H_n of continuous measures, such that if X is random for some continuous measure, then it is random relative to some μ in one of the H_n.

- n indicates the place after which g_μ is dominated by c_X.

Exotic examples

Observation

If the approximation to X converges quickly on intervals which are long compared to the speed of convergence on the earlier part of X, then X cannot be continuously random.

This direct characterization of NCR_1 on Δ^0_2 is compatible with other constructions and thus enables us to find exotic elements of NCR_1:

- 1-generic
- packing dimension 1

But if a 1-generic X is a path through a recursive tree, the tree must contain a whole basic cylinder $\mathrm{N}(X\restriction_n)$.