The Structure of NCR **inside** Δ^0_2

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(joint work with Theodore Slaman)

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Measures on Cantor Space

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- We require $\mu[\emptyset] = 1$ and $\mu[\sigma] = \mu[\sigma^\frown 0] + \mu[\sigma^\frown 1]$.
- If $\mu\{X\} > 0$ for $X \in 2^{\mathbb{N}}$, i.e. if $\lim_n \mu[X\restriction_n] > 0$, then X is called an atom of μ. A non-atomic measure is called continuous.

Measures on Cantor Space

Representation of measures

- The space $\mathcal{M}(2^{\mathbb{N}})$ of all probability measures on $2^{\mathbb{N}}$ is compact Polish. Furthermore, there is a computable surjection $\pi : 2^{\mathbb{N}} \to \mathcal{M}(2^{\mathbb{N}})$.
  A compatible metric is given by $d(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} d_n(\mu, \nu)$, where
  $$d_n(\mu, \nu) = \frac{1}{2} \sum_{|\sigma|=n} |\mu[\sigma] - \nu[\sigma]|.$$

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- One can also code a measure directly into a real R_μ by recording for each σ the rational intervals in which $\mu[\sigma]$ falls:
  $$(\sigma, q_0, q_1) \in R_\mu \quad \Longleftrightarrow \quad q_0 < \mu[\sigma] < q_1.$$
  However, this representation does not have the nice topological properties as above.

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- A μ-Z-test is a set $W \subseteq \mathbb{N} \times 2^{<\mathbb{N}}$ which is c.e. (Σ^0_1) in $r_\mu \oplus Z$ such that
$$\sum_{\sigma \in W_n} \mu([\sigma]) \leqslant 2^{-n},$$
where $W_n = \{\sigma : (n, \sigma) \in W\}$.

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Clearly, a real X is trivially μ-random if it is a μ-atom.

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- Any non-recursive real is non-trivially random with respect to some measure [Reimann and Slaman]. However, the measure may have atoms.
- All except for countably many reals are random with respect to a continuous measure. The non-continuously random reals form a subset of Δ^1_1. [Reimann and Slaman]

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  The proof yields that NCR $\subset \Delta^1_1$.

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- The result can be generalized to continuous n-randomness (tests have access to $\emptyset^{(n-1)}$). It holds that $\text{NCR}_n \subseteq L_{\beta_n}$, where β_n is least such that $L_{\beta_n} \models \text{ZFC}^-_n$.

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- Has an interesting metamathematical twist.

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Understand the structure of NCR *(inside* Δ^1_1*).*

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- Which known properties imply being in NCR?
- Which properties are compatible with being NCR?

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- If μ is continuous, given n, we can, for every rational $\varepsilon > 0$, compute (recursively in μ) a number $l(\varepsilon)$ such that

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- Enumerate $S_{n_1} \upharpoonright_{n_0}$ into V_n, and for all k such that $s_k < n_0$, enumerate the $(S_{n_1} \upharpoonright_{s_k}) ^\frown 0^{n_1 - s_k}$ into V_n.

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- Then (V_n) covers S.

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Elements of countable Π^0_1*-classes are also referred to as ranked points.*

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All ranked points are hyperarithmetic. Ranked points are cofinal in the hyperarithmetic Turing degrees: For every $Y \in \Delta^1_1$ there exists a ranked $X \geqslant_T Y$.

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Kreisel's analysis was based on the Cantor-Bendixson rank and the boundedness principle.

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- Canonical rank for Π^1_1 sets is derived from tree representation. However, this rank may not be very informative.
- The Kjos-Hanssen-Montalban result, combined with the fact that $\text{NCR} \subseteq \Delta^1_1$, suggests the Cantor-Bendixson rank for NCR.

Δ^0_2 Sets

The Δ^0_2 sets are sufficiently concrete so we can analyze their behavior regarding NCR directly.

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Clearly, whenever you can compute the settling function, you can compute X.

The Granularity Function

Definition

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Since for any measure,

$$\max\{\mu[\sigma^\frown 0], \mu[\sigma^\frown 1]\} \geqslant 1/2\mu[\sigma]$$

it follows that for all μ,

$$g_\mu(n) \geqslant n.$$

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Theorem
If $X \in 2^{\mathbb{N}}$ is Δ^0_2 and random with respect to μ, then c_X dominates g_μ, i.e. $c_X(n) > g_\mu(n)$ for all but finitely many n.

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> *This makes* NCR *compatible with other construction methods.*
>
> *We can use it to construct examples in* NCR *with additional properties.*

Specific Examples

Proposition

There exist elements in $\mathrm{NCR} \cap \Delta^0_2$ which are additionally

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The existence of 1-generic reals in NCR also shows that the Cantor-Bendixson rank is unsuitable for NCR.

> *If a* 1*-generic is an element of a* Π_1^0*-class* $[T]$*, then* $[T]$ *must contain a cylinder* $[\sigma]$*.*

NCR **and** K**-Triviality**

Montalban and Slaman applied the domination technique to show that a K-trivial is never continuously random.

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If X is such that

$$\exists c \, \forall n \;\; K(X\restriction_n) \leqslant K(0\restriction_n) + c,$$

then $X \in \mathrm{NCR}$*.*

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- Assume not, i.e. X is μ-random for some continuous μ.

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- Then the function $h(k) =$ (code of) $X\restriction_{f(k)}$ is diagonally non-recursive (dnr) (up to finitely many k).
- Hence Y computes a dnr function, which is impossible according to the Arslanov completeness criterion.

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((Theorem))
There is a Δ^0_2 set $X <_T \emptyset'$ such that $X \in$ NCR and X is not recursive in any incomplete r.e. set.

The Descriptive Complexity of $\mathrm{NCR} \cap \Delta_2^0$

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Theorem

For each Δ^0_2 set X, there is an arithmetically defined sequence of compact sets H_n of continuous measures, parametrized by the place after which g_μ is dominated by m_X, such that if X is 1-random relative to some continuous measure, then it is 1-random relative to some μ in one of the H_n.

The Descriptive Complexity of $\mathsf{NCR} \cap \Delta^0_2$

Finally, we can use the domination property to analyze the descriptive complexity of $\mathsf{NCR} \cap \Delta^0_2$.

Theorem

For each Δ^0_2 set X, there is an arithmetically defined sequence of compact sets H_n of continuous measures, parametrized by the place after which g_μ is dominated by m_X, such that if X is 1-random relative to some continuous measure, then it is 1-random relative to some μ in one of the H_n.

It follows that $\mathsf{NCR} \cap \Delta^0_2$ is an arithmetic set of reals.

Further Directions

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Some ideas:

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- Fourier coefficients: Replace the analysis of the granularity function by a growth analysis of Fourier coeffients. There are a number of descriptive set theoretic tools available for this.

Ende