Measures and Their Random Reals

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Algorithmic Randomness

Investigates individual random objects. Objects are usually infinite binary sequences (reals).

- Randomness: Obey statistical laws.
- Algorithmic: Only effective laws. (There are only countably many, so their intersection describes an almost sure event, hence random objects exist.)

Randomness and Computability

Recently a lot of progress in understanding the relation between two kinds of complexities for reals:

information theoretic	computability theoretic
randomness properties	degrees of unsolvability

Schnorr's Theorem

A real x is Martin-Löf random (with respect to the uniform distribution) iff

$$(\forall n)\ K(x\restriction_n) \geq^+ n,$$

where K denotes prefix-free Kolmogorov complexity.

However, investigations mostly fixed the underlying measure, Lebesgue measure, and studied different notions of randomness by varying the effectiveness conditions.

Question

What is the relation between logical and measure theoretic complexity if one allows arbitrary (continuous) probability measures?

The answer to this question took an unexpected turn.

Measures and cylinders

(Borel) probability measures are uniquely determined by their values on basic clopen cylinders

$$N_\sigma := \{x \in 2^\omega : \sigma \subset x\}.$$

where $\sigma \in 2^{<\omega}$.

Representation of measures

The space $\mathcal{P}(2^\omega)$ of all probability measures on 2^ω is compact Polish. Furthermore, there is a computable surjection

$$\pi : 2^\omega \to \mathcal{P}(2^\omega).$$

Effective Randomness

Effective G_δ sets

A test for randomness is an effectively presented G_δ nullset.

Definition

Let μ be a probability measure on 2^ω.

- A μ-test relative to $z \in 2^\omega$ is a set $W \subseteq \mathbb{N} \times 2^{<\omega}$ which is c.e. (Σ^0_1) in z such that

  $$\sum_{\sigma \in W_n} \mu(N_\sigma) \leq 2^{-n},$$

  where $W_n = \{\sigma : (n, \sigma) \in W\}$
- A real x passes a test W if $x \notin \bigcap_n N(W_n)$, i.e. if it is not in the G_δ-set represented by W.

Definition

Suppose μ is a measure and $z \in 2^\omega$. A real x is μ-random relative to z if there exists a representation ρ_μ of μ such that x passes all μ-tests relative to $\rho_\mu \oplus z$.

- n-randomness: tests c.e. in $\rho_\mu^{(n-1)}$.
- Accordingly, define arithmetical randomness.

Trivial Randomness

Obviously, every real x is trivially random with respect to μ if $\mu(\{x\}) > 0$, i.e. if x is an atom of μ.

If we rule out trivial randomness, then being random means being non-computable.

Theorem

For any real x, the following are equivalent.

- *There exists a measure μ such that $\mu(\{x\}) = 0$ and x is μ-random.*
- *x is not computable.*

Features of the proof

- Conservation of randomness.
  If y is random for Lebesgue measure $\mathcal{L}$, and $f : 2^\omega \to 2^\omega$ is computable, then $f(y)$ is random for $\mathcal{L}_f$, the image measure.
- A cone of $\mathcal{L}$-random reals.
  By the Kucera-Gacs Theorem, every real above $0'$ is Turing equivalent to a $\mathcal{L}$-random real.
- Relativization using the Posner-Robinson Theorem.
  If a real is not computable, then it is above the jump relative to some G.
- A compactness argument for measures measures.

Randomness for Continuous Measures

In the proof we have little control over the measure that makes x random.

- In particular, atoms cannot be avoided (due to the use of Turing reducibilities).

Question

What if one admits only continuous (i.e. non-atomic) probability measures?.

The Class NCR

Let NCR_n be the set of all reals which are not n-random with respect to any continuous measure.

Question

What is the structure/size of NCR_n*?*

- *Is there a level of logical complexity that guarantees continuous randomness?*
- *Can we reproduce the proof that a non-computable real is random at a higher level?*

Easy upper bound

NCR_n is a Π^1_1 set.

- NCR_n does not have a perfect subset.
- Solovay, Mansfield: Every Π^1_1 set of reals without a perfect subset must be contained in L.

Randomness for Continuous Measures

Characterizing randomness for continuous measures

One can analyze the proof of the previous theorem to obtain a more recursion theoretic characterization of continuous randomness.

Theorem

Let x be a real. For any $z \in 2^\omega$, the following are equivalent.

- *x is random for a continuous measure computable in z.*
- *There exists a functional Φ computable in z which is an order-preserving homeomorphism of 2^ω such that $\Phi(x)$ is $\mathcal{L}$-z-random.*
- *x is truth-table equivalent (relative to z) to a $\mathcal{L}$-z-random real.*

This is an effective version of the classical isomorphism theorem for continuous probability measures.

An upper cone of continuously random reals

- Show that the complement of NCR_n contains a Turing invariant and cofinal (in the Turing degrees) Borel set.
- We can use the set of all x that are Turing equivalent to some $z \oplus R$, where R is $(n+1)$-random relative to a given z.
- These x will be n-random relative to some continuous measure and are T-above z.
- Use Martin's result on Borel Turing determinacy to infer that the complement of NCR_n contains a cone.
- The base of the cone is given by the Turing degree of a winning strategy in the corresponding game.

Martin's proof is constructive

- The direct nature of Martin's proof implies that the winning strategy for that game belongs to the smallest L_β such that L_β is a model of (a sufficiently large subset of) ZFC (plus relativization).
- The more complicated the game is in the Borel hierarchy, the more iterates of the power set of the continuum are used in producing the winning strategy – trees, trees of trees, etc.
- More precisely, the winning strategy (for Borel complexity n) is contained in

$$L_{\beta_n} \models \mathrm{ZFC}_n^-$$

where ZFC_n^- is Zermelo-Fraenkel set theory without the Power Set Axiom + "there exist n many iterates of the power set of $\mathcal{P}(\omega)$".

Posner-Robinson-style relativization

- Given $x \notin L_{\beta_n}$, using forcing we construct a set G such that $L_{\beta_n}[G] \models \mathrm{ZFC}_n^-$ and

$$y \in L_{\beta_n}[G] \cap 2^\omega \quad \text{implies} \quad y \leq_{\mathrm{T}} x \oplus G$$

  (independently by Woodin).
- If x is not in L_{β_n}, it will belong to every cone with base in the accordant $L_{\beta_n}[G]$, in particular, it will belong to the cone avoiding NCR_n. (Use absoluteness)

Corollary

For all n, NCR_n is countable.

NCR_n is Countable

Metamathematics necessary?

Question

Do we really need the existence of iterates of the power set of the reals to prove the countability of NCR_n*, a set of reals?*

We make fundamental use of Borel determinacy; this suggests to analyze the metamathematics in this context.

Necessity of power sets – Friedman's result

- Friedman showed

$$\mathrm{ZFC}^- \nvdash \boldsymbol{\Sigma}^0_5\text{-determinacy}.$$

  (Martin improved this to $\boldsymbol{\Sigma}^0_4$.)
- The proof works by showing that there is a model of ZFC^- for which $\boldsymbol{\Sigma}^0_4$-determinacy does not hold. This model is L_{β_0}.

NCR and Iterates of the Power Set

We can proof a similar result concerning the countability of NCR_n.

Theorem

For every k,

$$\mathsf{ZFC}^-_k \nvdash \text{“For every } n, \mathsf{NCR}_n \text{ is countable”}.$$

NCR and Iterates of the Power Set

Features of the proof

NCR_n is not countable in L_{β_0}

- Show that there is an n such that NCR_n is cofinal in the Turing degrees of L_{β_0}. (The approach does not change essentially for higher k.)
- The non-random witnesses will be the reals which code the full inductive constructions of the initial segments of L_{β_0}.

Randomness does not accelerate defining reals

Suppose that $n \geq 2$, $y \in 2^\omega$, and x is n-random for μ. Then, for $i < n$,

$$y \leq_T x \oplus \mu \text{ and } y \leq_T \mu^{(i)} \quad \text{implies} \quad y \leq_T \mu.$$

NCR and Iterates of the Power Set

Features of the proof

Example

For all k, $0^{(k)}$ is not 3-random for any μ.

Proof

- Suppose $0^{(k)}$ is 3-random relative to μ.
- $0'$ is computably enumerable relative to μ and computable in the supposedly 3-random $0^{(k)}$. Hence, $0'$ is computable in μ and so $0''$ is computablely enumerable relative to μ.
- Use induction to conclude $0^{(k)}$ is computable in μ, a contradiction.

NCR and Iterates of the Power Set

L_α's and their master codes

Master codes

- L_α, $\alpha < \beta_0$, is a countable structure obtained by iterating first order definability over smaller L_γ's and taking unions.
- Jensen's master codes are a sequence $M_\alpha \in 2^\omega \cap L_{\beta_0}$, for $\alpha < \beta_0$, of representations of these countable structures.
- M_α is obtained from smaller M_γ's by iterating the Turing jump and taking arithmetically definable limits.
- Every $x \in 2^\omega \cap L_{\beta_0}$ is computable in some M_α.

Master codes as witnesses for NCR

- An inductive argument similar to the example $0^{(k)} \in \mathrm{NCR}_3$ can be applied transfinitely to these master-codes.
- There is an n such that for all limit λ, if $\lambda < \beta_0$ then $M_\beta \in \mathrm{NCR}_n$.

The Structure of NCR_1

Question

What is the structure of NCR_1?

NCR_1 and Δ^1_1

By analyzing the complexity of a the winning strategy for (effectively) closed games we obtain that every member of NCR_1 is hyperarithmetic.

Countable Π^0_1 classes

- Kjos-Hanssen and Montalban: Every member of a countable Π^0_1 class is contained in NCR_1.
- It follows that NCR_1 is cofinal in the hyperarithmetical Turing degrees. (Kreisel, Cenzer et al.)

The Structure of NCR_1

Looking for a rank function

The Kjos-Hanssen-Montalban result suggests that the complexity of NCR_1 could be studied by means of a Cantor-Bendixson analysis.

However, this is not possible:

Theorem

There exists an $x \in \mathsf{NCR}_1$ *that is not a member of any countable* Π^0_1 *class.*

The Structure of NCR_1

Non-ranked examples

Lemma 1

If a computable tree T does not contain a computable path, then no member of $[T]$ can be an element of a countable Π^0_1 set.

Lemma 2

There exists a computable tree T such that T has no computable path and for all $\sigma \in T_\infty$, if there exist n branches along σ, then $0' \upharpoonright_n$ is settled by stage $|\sigma|$.

Lemma 3

If a recusive tree T contains a μ-random path, then $\mu[T] > 0$.

The Structure of NCR_1

Non-ranked examples

Proof of the Theorem

- Suppose every infinite path in T is continuously random.
- Let x be a Δ^0_2 path in T. Suppose x is μ-random.
- Recursively in μ, we can compute a function $h : \mathbb{N} \to \mathbb{N}$ such that some element in $[T]$ must have n-many branchings in T_∞ by level $h(n)$.
- Hence, by construction of T, μ computes $0'$, hence computes x, contradiction!

The Structure of NCR_1

Δ^0_2 reals

We can exploit the splitting behavior of continuous measures further to obtain more information of Δ^0_2 members of NCR_1.

Settling and splitting

- Let x be Δ^0_2 and let $c_x : \omega \to \omega$ be defined by

  $$c_x(n) = \min\{s :\ x(n) \text{ is settled by stage } s\}$$

  x can be computed from any function g which dominates c_x pointwise.
- When μ is a continuous measure, we can extract a granularity function $g_\mu : \omega \to \omega$ with the following property:

  $$\text{For all } \sigma \text{ of length } g_\mu(n),\ \mu([\sigma]) < 1/2^n.$$

Dominating the settling function

- If g_μ dominates c_x pointwise, then x is recursive in μ and hence not μ-random.
- An argument along this line shows, if g_μ is not eventually dominated by c_x, then x can be approximated in measure and is not μ-random.

Theorem

For each Δ^0_2 x, there is an arithmetically defined sequence of compact sets H_n of continuous measures, such that if x is random for some continuous measure, then it is random for some μ in one of the H_n.

The Structure of NCR_1

Other examples

This technique can be used to obtain examples in NCR_1

- Δ^0_2 and sufficiently generic,
- of minimal degree.
- of packing dimension 1.

On the other hand, reals cannot be in NCR_1 if they have a computable nontrivial lower bound on their Kolmogorov complexity.