Schnorr Dimension

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Why Effectivize Measure

- Can explicitly consider **typical** elements (with respect to measure).
- Allows to define **random** elements.
- Can apply measure theory to **countable** sets/spaces.

Ways to Effectivize Measure

Effectivizing Measure $\hat{=}$ devising an **effective** class of **tests**. Each test determines a class of nullsets.

- **Martin-Löf**: Tests must be **effectively** G_δ.
- **Schnorr**: Test must have **uniformly computable** measure.
- **Martingales** (Schnorr/Lutz): Nullsets are those against which a **computable** martingale wins.
- **Semimeasures/complexity**: Elements of nullsets must be **compressible**.

Hausdorff Measures

Definition

- Given $s > 0$, $\mathcal{A} \subseteq \{0,1\}^{\mathbb{N}}$ has ***s*-dimensional Hausdorff measure** 0, $\mathcal{H}^s(\mathcal{A}) = 0$, if for all n there exists $C_n \subseteq \{0,1\}^*$ such that

$$\mathcal{A} \subseteq \bigcup_{\sigma \in C_n} \mathrm{Ext}(\sigma) \ \wedge \ \sum_{\sigma \in C_n} 2^{-|\sigma| s} \leqslant 2^{-n}.$$

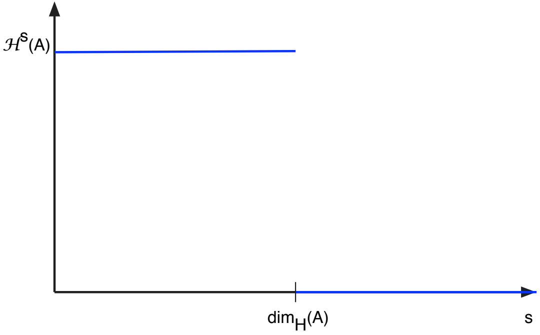
- So for $s = 1$, one obtains **Lebesgue measure** on $\{0,1\}^{\mathbb{N}}$.

Hausdorff Dimension

Definition

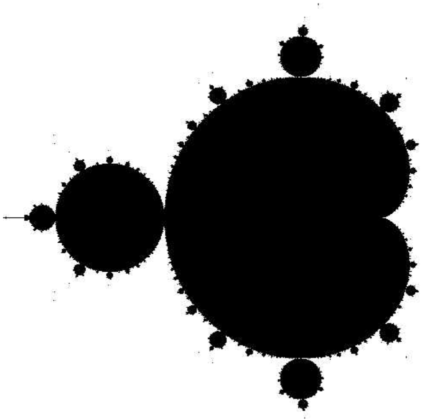
The **Hausdorff dimension** of A is defined as

$$\dim_H(A) = \inf\{s \geqslant 0 : \mathcal{H}^s(A) = 0\}$$



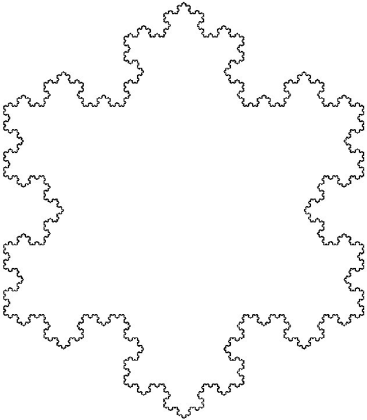
Famous examples

Mandelbrot sets – $\dim_H = 2$



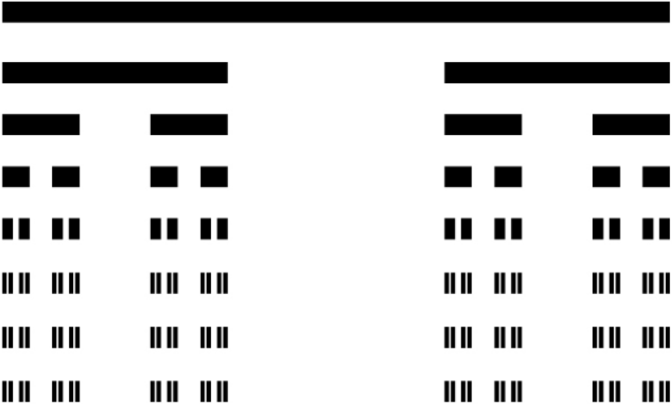
Famous examples

Koch snowflake – $\dim_H = \log 4 / \log 3$



Famous examples

Cantor set – $\dim_H = \log 2 / \log 3$



Effective Hausdorff Measures

Definition

Let $s \geqslant 0$ be rational.

- A **Martin-Löf s-test** (ML-s-test) is a uniformly computable sequence $(V_n)_{n \in \mathbb{N}}$ of c.e. sets of strings such that for all n,

$$\sum_{\sigma \in V_n} 2^{-|\sigma| s} \leqslant 2^{-n}.$$

- A test (V_n) **covers** a real X if $X \in \bigcap_n \operatorname{Ext}(V_n)$
- X is **ML-s-random** if it is not covered by ML-s-test.
- A **Schnorr s-test** is a ML-s-test (V_n) such that the real number $\sum_{\sigma \in V_n} 2^{-|\sigma| s}$ is uniformly computable.
- X is **Schnorr-s-random** if it is not covered by Schnorr-s-test.

Effective Hausdorff Dimension

We can now easily define effective versions of Hausdorff dimension. These can be considered as **degrees of randomness**.

Definition

Let X be a real.

- (Lutz) The **effective Hausdorff dimension** $\dim_{\mathrm{H}}^{1} X$ is defined as

$$\dim_{\mathrm{H}}^{1} X = \inf\{s \in \mathbb{Q}^{+} : \{X\} \text{ is covered by a ML-}s\text{-test}\}.$$

- The **Schnorr Hausdorff dimension** $\dim_{\mathrm{H}}^{\mathrm{S}} X$ is defined as

$$\dim_{\mathrm{H}}^{\mathrm{S}} X = \inf\{s \in \mathbb{Q}^{+} : \{X\} \text{ is covered by a Schnorr-}s\text{-test}\}.$$

Martingales and Computable Randomness

- A **martingale** is a function $d : \{0,1\}^* \to \mathbb{R}_0^+$ such that for all strings σ,
$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}.$$
- For $s \geqslant 0$, a martingale is **s-successful** on a real X if $\limsup_n d(X \restriction_n)/2^{(1-s)n} = \infty$.
- A real X is **computably s-random** if no computable martingale d is s-successful on X.
- Known: Computably s-random $\Rightarrow$ Schnorr s-random. But there are Schnorr 1-random sequences not computably 1-random (Wang).

Dimension and Martingales

Theorem

For any real $X \in \{0,1\}^{\mathbb{N}}$,

$$\dim_{\mathrm{H}}^{\mathrm{S}} X = \inf\{s \in \mathbb{Q} : \exists \textbf{ computable } d \ s\textbf{-succ. on } X\}.$$

So for Schnorr Hausdorff dimension it does not matter whether one works with computable martingales or Schnorr tests.
Schnorr dimension equals computable dimension.

Machine Characterizations

- Given a (prefix-free) Turing machine M , the ***M*-complexity** of a string x is defined as

$$\mathrm{K}_M(x) = \min\{|p| : \; M(p) = x\},$$

where $\mathrm{K}_M(x) = \infty$ if there does not exist a $p \in \{0,1\}^*$ such that $M(p) = x$.

- For a universal prefix-free TM U, $\mathrm{K} := \mathrm{K}_U$ is **optimal up to a fixed constant**, i.e. for all prefix-free M exists c_M s.t. $\forall x(\mathrm{K}(x) \leqslant \mathrm{K}_M(x) + c_M)$.
- The effective dimension of a real equals its **lower asymptotic complexity**:

$$\dim^1_{\mathrm{H}} X = \liminf_n \frac{\mathrm{K}(X\restriction_n)}{n}.$$

(Shown independently by Ryabko and Mayordomo.)

Machine Characterizations

Call a prefix free machine M is **computable** if $\sum_{w \in \mathrm{dom}(M)} 2^{-|w|}$ is a computable real number.

Theorem

For any sequence A it holds that

$$\dim_{\mathrm{H}}^{\mathrm{S}} A = \inf_{M} \left\{ \liminf_{n \to \infty} \frac{\mathrm{K}_M(A\restriction_n)}{n} \right\},$$

where the infimum is taken over all computable prefix free machines M.

A similar characterization was obtained by Hitchcock.

Packing Dimension

- **Packing measures** (Tricot) are dual to Hausdorff measures: Instead of covering a set with as few balls as possible, try to 'stuff' it with as many disjoint balls as possible.
- The corresponding dimension notion, **Packing dimension** $\dim_{\mathrm{P}}$, can be effectivized ($\dim^1_{\mathrm{P}}$) using a martingale characterization (Athreya, Hitchcock, Lutz, and Mayordomo).
- The effective packing dimension of a real equals its **upper asymptotic complexity** (Athreya et al):

$$\dim^1_{\mathrm{P}} X = \limsup_n \frac{\mathrm{K}(X\restriction_n)}{n}.$$

- **Schnorr version**:

$$\dim^{\mathrm{S}}_{\mathrm{P}} A := \inf_M \left\{ \limsup_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n} \right\}.$$

Recursively Enumerable Sets

- The main randomness notions (Martin-Löf, computable, and Schnorr) are powerful enough to render r.e. sets trivial, i.e. **no r.e. set is random**.
- In fact, they are not even close to random: For any r.e. set $A \subseteq \mathbb{N}$,

$$K(A\restriction_n) \leqslant k \log n + c.$$

(Barzdins' Theorem)

- With respect to Schnorr dimension, the situation is a little different.

Theorem

1. *Every r.e. set $A \subseteq \mathbb{N}$ has Schnorr Hausdorff dimension zero.*
2. *There exists an r.e. set $A \subseteq \mathbb{N}$ such that $\dim^{S}_{P} A = 1$.*

R.E. Sets And Irregularity

Theorem

1. *Every r.e. set* $A \subseteq \mathbb{N}$ *has Schnorr Hausdorff dimension zero.*
2. *There exists an r.e. set* $A \subseteq \mathbb{N}$ *such that* $\dim_{\mathrm{P}}^{\mathrm{S}} A = 1$.

- Tricot defined a set to be **regular** if its Hausdorff and packing dimension coincide.
- Hence, the class of r.e. sets contains **examples of irregular reals** with respect to Schnorr dimension.