Hausdorff Dimension, Randomness, and Entropy

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Hausdorff measures

- Caratheodory-Hausdorff construction on metric spaces: X metric space $E \subseteq X$, metric d, $h : \mathbb{R} \to \mathbb{R}$ non-decreasing, continuous on the right with $h(0) = 0$, $\delta > 0$.

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- Define set function

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum_i h(d(U_i)) : \; E \subseteq \bigcup_i U_i, \; d(U_i) \leq \delta \right\}.$$

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- Letting $\delta \to 0$ yields an (outer) measure.
- The h-dimensional Hausdorff measure $\mathcal{H}^h$ is defined as

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(E)$$

Properties of Hausdorff measures

- $\mathcal{H}^h$ is Borel regular:
  all Borel sets B are measurable, i.e.

$$(\forall A \subseteq X)\, \mathcal{H}^h(A) = \mathcal{H}^h(A \cap B) + \mathcal{H}^h(A \setminus B),$$

  and for all $A \subseteq X$ there is a Borel set $B \subseteq A$ such that

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- For $X = \mathbb{R}^n$ (Euclidean) and $s = n$, $\mathcal{H}^n$ yields the usual Lebesgue measure λ (up to a multiplicative constant).

From measure to dimension

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- Important property: For $0 \leq s < t < \infty$ und $E \subseteq X$,

$$\mathcal{H}^s(E) < \infty \;\Rightarrow\; \mathcal{H}^t(E) = 0,$$
$$\mathcal{H}^t(E) > 0 \;\Rightarrow\; \mathcal{H}^s(E) = \infty.$$

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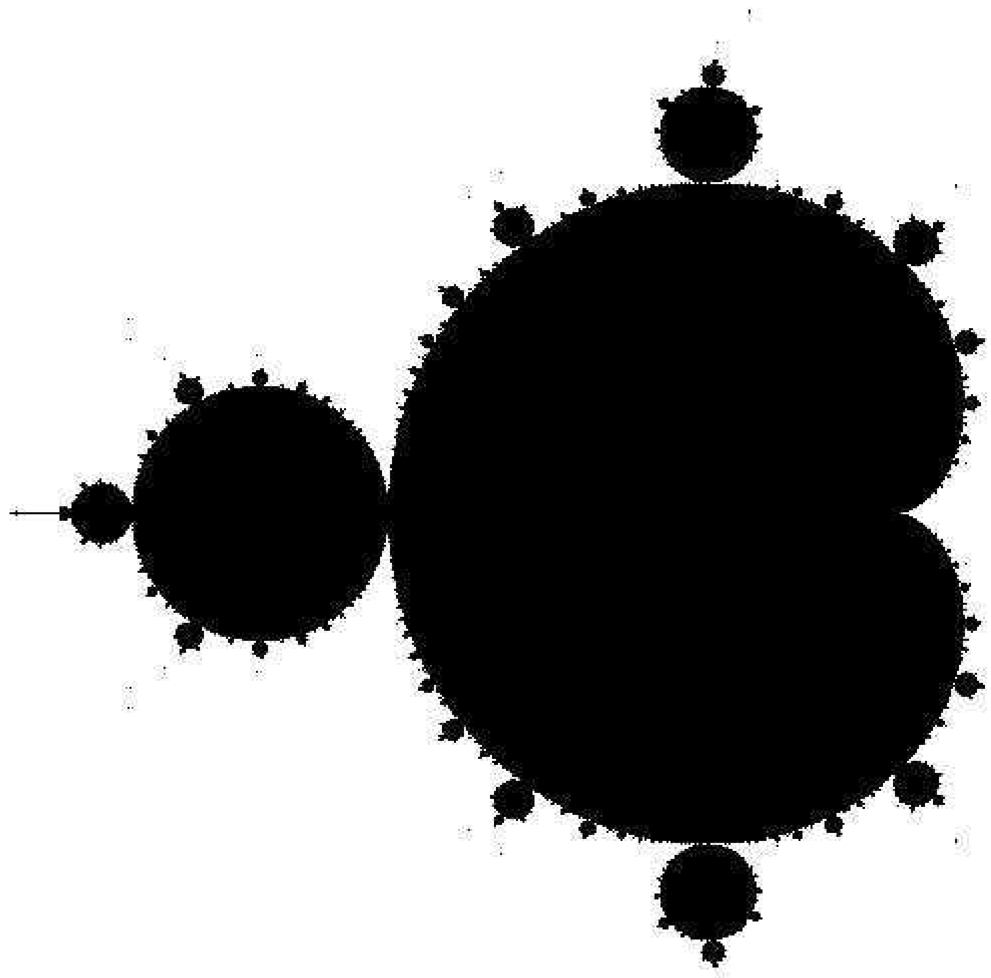
- The Hausdorff dimension of a set E is defined as

$$\begin{aligned} \dim_{\mathrm{H}}(E) &= \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} \\ &= \sup\{t \geq 0 : \mathcal{H}^t(E) = \infty\} \end{aligned}$$

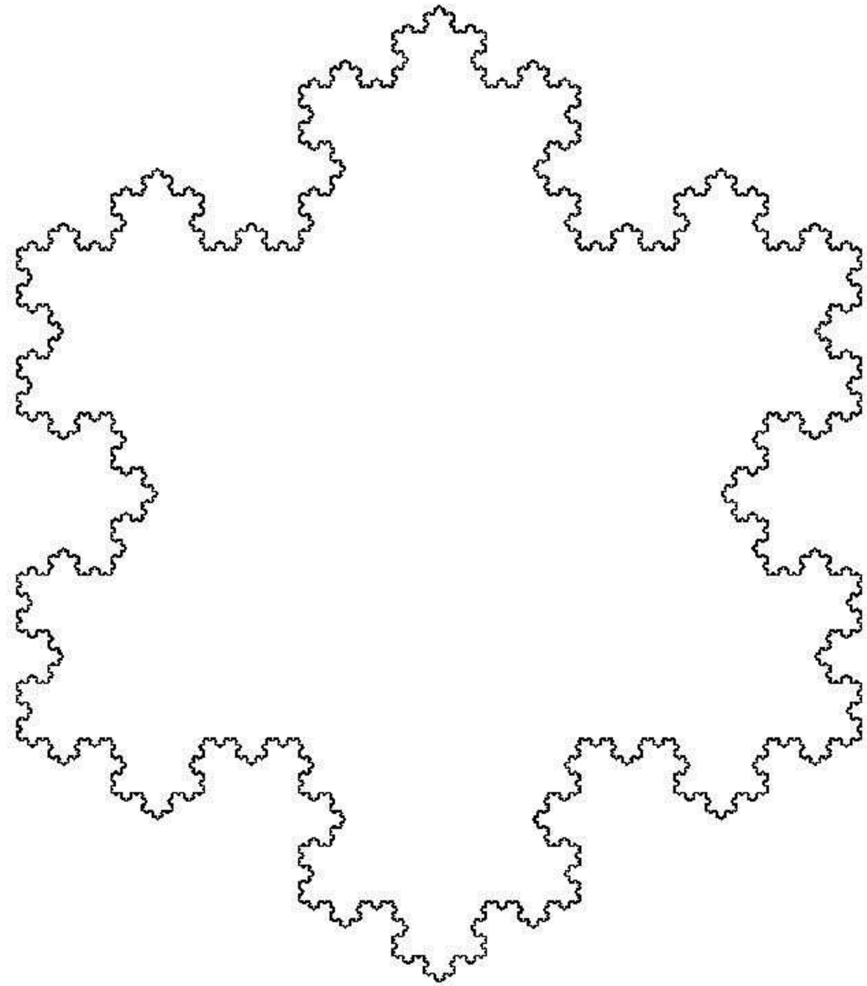
Famous examples

Mandelbrot sets – $\dim_H = 2$



Famous examples

Koch snowflake – $\dim_H = \log 4 / \log 3$



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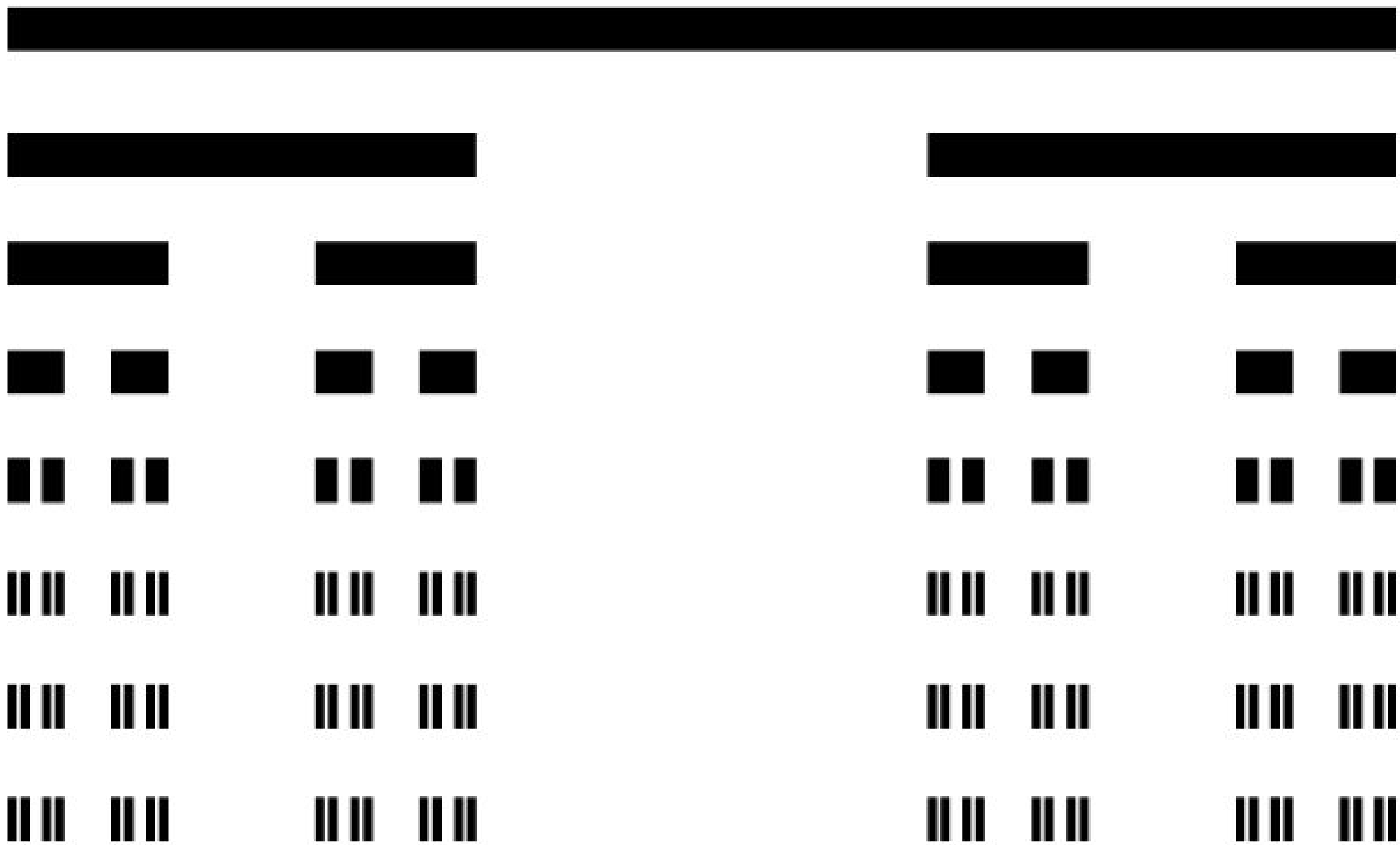
Cantor set – $\dim_H = \log 2 / \log 3$



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- Open sets in Cantor space are unions of cylinders, induced by a finite string $\sigma \in 2^{<\omega}$.

$$[\sigma] := \{\alpha : \sigma \sqsubset \alpha\}.$$

Diameter $d[\sigma] = 2^{-|\sigma|}$.

Hausdorff dimension in Cantor space

- $\mathcal{H}^s$-nullsets in 2^ω:

  $\mathcal{A} \subseteq 2^\omega$ has s-dimensional Hausdorff measure 0 iff

$$(\forall n \in \omega)\ (\exists C_n \subseteq 2^{<\omega})\ \mathcal{A} \subseteq \bigcup_{\sigma \in C_n} [\sigma]\ \wedge\ \sum_{\sigma \in C_n} 2^{-|\sigma|s} \leq 2^{-n}.$$

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- Effectivization: Require C_n to be effectively given, e.g. as a uniformly recursive family of r.e. sets of strings.

An example from recursion theory

- Theorem: [Sacks]
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- Theorem:
  $$(\forall \alpha \in 2^\omega)\ \dim_{\mathrm{H}}(\alpha^{\leq_{\mathrm{T}}}) = 1.$$
- Mass distribution principle: $\mathcal{A} \subseteq 2^\omega$, μ measure on 2^ω, $\mu(\mathcal{A}) > 0$. If there are c, s such that
  $$\mu[\sigma] \leq c 2^{-|\sigma| s} = c d[\sigma]^s$$
  (for all σ), then $\dim_{\mathrm{H}} \mathcal{A} \geq s$.

Properties of Hausdorff dimension

- Lebesgue measure: $\lambda(\mathcal{A}) > 0$ implies $\dim_{\mathrm{H}}(\mathcal{A}) = 1$.

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- Monotony: $\mathcal{A} \subseteq \mathcal{B}$ implies $\dim_{\mathrm{H}}(\mathcal{A}) \leq \dim_{\mathrm{H}}(\mathcal{B})$.
- Stability: For $\mathcal{A}_1, \mathcal{A}_2, \dots \subseteq 2^\omega$ it holds that

$$\dim_{\mathrm{H}}(\bigcup \mathcal{A}_i) = \sup\{\dim_{\mathrm{H}}(\mathcal{A}_i)\}.$$

(Immediately implies that all countable sets have dimension 0.)

Properties of Hausdorff dimension

- Geometric transformations: If h is Hölder continuous, i.e. if there are constants $c, r > 0$ for which

$$(\forall \alpha, \beta)\ d(h(\alpha), h(\beta)) \leq c d(\alpha, \beta)^r,$$

then

$$\dim_{\mathrm{H}} h(\mathcal{A}) \leq (1/r) \dim_{\mathrm{H}}(\mathcal{A}).$$

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- Fractal geometry $\hat{=}$ study properties invariant under bi-Lipschitz trandformations.

Dimension and entropy

- For $\delta = 2^{-n}$, simple δ-covering for $\mathcal{A} \subseteq 2^\omega$:

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- Minkowski- or box-counting dimension:

$$\underline{\dim}_{\mathrm{B}}(\mathcal{A}) := \liminf_{n \to \infty} \frac{\log |A^{[n]}|}{n}.$$

It holds that $\dim_{\mathrm{H}}(\mathcal{A}) \leq \underline{\dim}_{\mathrm{B}}(\mathcal{A})$.

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- If $\mathcal{A}$ is shift-invariant, $\underline{\dim}_{\mathrm{B}}$ is also called topological entropy, and for closed sets $\mathcal{A}$ it holds that

$$\dim_{\mathrm{H}}(\mathcal{A}) = \underline{\dim}_{\mathrm{B}}(\mathcal{A}).$$

Dimension and entropy

- Given $p \in [0, 1]$, let μ_p $(p, 1 - p)$-Bernoulli measure (product measure on 2^ω with $P[1] = p$, $P[0] = 1 - p$). Entropy $H(\mu_p)$ is defined as

$$H(\mu_p) = -[p \log p + (1 - p) \log(1 - p)].$$

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$$H(\mu_p) = -[p \log p + (1-p) \log(1-p)].$$

- Theorem: [Eggleston] For $p \in [0,1]$ let

$$\mathcal{B} = \left\{ \alpha \in 2^\omega : \lim_{n \to \infty} \frac{|\{i \leq n : \alpha(i) = 1\}|}{n} = p \right\}.$$

Then

$$\dim_{\mathrm{H}} \mathcal{B} = H(\mu_p).$$

Effective Hausdorff dimension

- Introduce effective coverings and define the notion of effective $\mathcal{H}^s$-measure 0.

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- (Let $s \geq 0$ be rational.) $\mathcal{A} \subseteq 2^\omega$ is Σ_1-$\mathcal{H}^s$ null, Σ^0_1-$\mathcal{H}^s(\mathcal{A}) = 0$, if there is a recursive sequence (C_n) of r.e. sets such that for each n,

$$\mathcal{A} \subseteq \bigcup_{w \in C_n} [w] \quad \text{and} \quad \sum_{w \in C_n} 2^{-|w|s} < 2^{-n}.$$

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- Definition of effective Hausdorff dimension is straightforward:

$$\dim^1_{\mathrm{H}}(\mathcal{A}) = \inf\{s \geq 0 : \ \Sigma^0_1\text{-}\mathcal{H}^s(\mathcal{A}) = 0\}.$$

Properties of effective dimension

- Monotony is conserved. Obviously, also $\dim_{\mathrm{H}} \mathcal{A} \le \dim^1_{\mathrm{H}} \mathcal{A}$.

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- Random sequences: Σ^0_1-$\mathcal{H}^1$ corresponds to Martin-Löf's effective measure. A sequence α which is not Σ^0_1-$\mathcal{H}^1$-null is called Martin-Löf-random.
  Obviously, if α is ML-random then $\dim^1_{\mathrm{H}} \alpha = 1$.

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  Obviously, if α is ML-random then $\dim_{\mathrm{H}}^1 \alpha = 1$.
- Stability: [Lutz] $\dim_{\mathrm{H}}^1 \mathcal{A} = \sup_{\xi \in \mathcal{A}} \dim_{\mathrm{H}}^1 \xi$
  Follows from existence of maximal effective s-nulltests, i.e. a recursive sequence of r.e. sets $\{U_n^s\}$, for which

$$\mathcal{A} \text{ is } \Sigma_1^0\text{-}\mathcal{H}^s\text{-null} \quad \Longleftrightarrow \quad (\forall \alpha \in \mathcal{A})\, \alpha \in \bigcap_n [U_n^s].$$

Algorithmic entropy

- Kolmogorov complexity: Let U be a universal Turing machine. For a string σ define

$$C(\sigma) = C_U(\sigma) = \min\{|p| : \ p \in 2^{<\omega}, \ U(p) = \sigma\},$$

i.e. $C(\sigma)$ is the length of the shortest U-program for σ. (Independent (up to an additiv constant) of the choice of U.)

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- Variant: prefix-free complexity K. Based on prefix-free Turing-machines – no two converging inputs are prefixes of one another.

Entropy and randomness

- Schnorr's Theorem:

$$\alpha \text{ ML-random} \quad \Leftrightarrow \quad (\exists c)\,(\forall n)\; K(\alpha \upharpoonright_n) \geq n - c.$$

Kolmogorov complexity and coding

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- Kraft-Chaitin Theorem: $\{\sigma_i\}_{i\in\mathbb{N}}$ set of strings, $\{l_i, l_2, \dots\}$ sequence of natural numbers ('lengths') such that

$$\sum_{i\in\mathbb{N}} 2^{-l_i} \leq 1,$$

then one can construct (primitive recursively) a prefix-free TM M and strings $\{\tau_i\}_{i\in\mathbb{N}}$, such that

$$|\tau_i| = l_i \quad \text{and} \quad M(\tau_i) = \sigma_i.$$

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- Coding Theorem: [Zvonkin-Levin]

$$K(\sigma) = -\log \widetilde{m}(\sigma) + c.$$

‘Main theorem’ of effective dimension

- Theorem: [Ryabko; Staiger; Mayordomo]
  For all $\xi \in 2^\omega$

$$\dim_{\mathrm{H}}^{1}(\xi) = \liminf_{n\to\infty} \frac{K(\xi\restriction_n)}{n}.$$

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- K can be replaced by C, since

$$C(\sigma) \leq K(\sigma) \leq C(\sigma) + 2\log C(\sigma).$$

An easy proof

- It holds that $\mathcal{A}$ has effective s-dim. Hausdorff measure 0 iff there exists a discrete semimeasure m enumerable from below such that for any $\alpha \in \mathcal{A}$,

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$$\dim^1_{\mathrm{H}}(\beta) < s \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \frac{\widetilde{m}(\beta\restriction_n)}{2^{-ns}} = \infty.$$

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- Using the Coding Theorem, this is equivalent to

$$(\overset{\infty}{\exists} n)[K(\beta \upharpoonright_n)/n < s].$$

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- C is r.e. and it holds that $\sum_{w\in C} 2^{-|w|s} \leq 1$.
- Extract an effective s-test for β from C by defining C_n to contain all strings w that have 2^n predecessors already enumerated by the time they are enumerated in C.

Two examples

- 'Diluted' randomness: ξ ML-random, define

$$\widehat{\xi} = \xi(0)\,0\,\xi(1)\,0\,\xi(2)\,0\ \ldots$$

Then $\dim^1_{\mathrm{H}} \widehat{\xi} = 1/2$. (Use K-complexity or Hölder property.)

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- Eggleston’s sequneces: μ_p $(p, 1-p)$-Bernoulli measure (p rational). Then, for all μ_p-random sequence ξ,

$$\dim^1_{\mathrm{H}} \xi = H(\mu_p).$$

Degrees and lower cones

- $Z \subseteq \mathbb{N}$ infinite, co-infinite, recursive.
  Def. $\alpha \oplus_Z \beta$: unique γ such that

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- Theorem: If $\lim_{n\to\infty} \frac{1}{n} |Z \cap \{0, \ldots, n-1\}| = \delta$, then

$$\dim_{\mathrm{H}}^1 \alpha \oplus_Z \beta \geq \delta \dim_{\mathrm{H}}^1 \alpha + (1-\delta) \dim_{\mathrm{H}}^{1,\alpha} \beta.$$

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- Theorem: For all $\alpha \in 2^\omega$, $\dim_{\mathrm{H}}^1(\alpha^{\equiv_{\mathrm{T}}}) = \dim_{\mathrm{H}}^1(^{\leq_{\mathrm{T}}} A)$

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- Corollary: $K^{\equiv_{\mathrm{tt}}}$ is ML-null but $\dim^1_{\mathrm{H}} K^{\equiv_{\mathrm{tt}}} = 1$.
- Theorem: $\dim^1_{\mathrm{H}} {}^{\leq_{\mathrm{btt}}}K = 0$ (and hence $\dim^1_{\mathrm{H}} K^{\equiv_{\mathrm{btt}}} = 0$).

A lower cone of dimension 0

- A sequence $\omega \in 2^{\omega}$ is 1-generic if for every r.e. set $U \subseteq 2^{<\omega}$ it holds that

$$(\exists x \sqsubset \omega)\,[x \in U \text{ or no extension of } x \text{ is in } U].$$

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$$(\exists x \sqsubset \omega)\ [x \in U \text{ or no extension of } x \text{ is in } U].$$

- Theorem: If $\xi \in 2^\omega$ is 1-generic, then

$$\beta \leq_{\mathrm{T}} \xi \quad \Rightarrow \quad \dim^1_{\mathrm{H}} \beta = 0,$$

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  which is equivalent to $\dim^1_{\mathrm{H}}(^{\leq_{\mathrm{T}}}\xi) = 0$.

- Proof uses result by Kucera and Demuth:
  If ω is 1-generic and $\beta \leq_{\mathrm{T}} \omega$, then any simple set $S \subseteq 2^{<\omega}$ contains a string w such that $w \sqsubset \beta$.

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- Theorem: Let $\mu_{\vec{p}}$, $\vec{p} = (p_0, p_1, \ldots)$, be a computable generalized Bernoulli measure with limit frequency p. If α is $\mu_{\vec{p}}$-random, then

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$$\dim^1_{\mathrm{H}}(^{\leq_{\mathrm{m}}}\alpha) = H(\mu_{\vec{p}}).$$

- Proof: We know that $\dim^1_{\mathrm{H}} \alpha = H(\mu_{\vec{p}})$.
  Furthermore, α is Kolmogorov-Loveland stochastic with respect to $\mu_{\vec{p}}$.

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- Theorem: [Levin]
  Every sequence which is random relative to some computable measure computes a Martin-Löf random sequence.
- If the measure is computable and non-atomic, the reduction is even truth-table.

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- Uses: If $\zeta \in 2^\omega$ is 1-generic, then, for any $\omega \in 2^\omega$ and any computable, infinite, co-infinite set $Z \subseteq \mathbb{N}$, $\zeta \oplus_Z \omega$ is not a natural sequence.

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(2) For every nondecreasing, nonconstant, recursive function g, there is an n such that

$$K(\alpha \upharpoonright_n) \leq g(n).$$

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- However, it turns out that arbitrary randomness is to ‘coarse’. It can only distinguish between recursive and non-recursive sequences.

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- Can we at least bound the complexity of a measure rendering a sequence of positive dimension random?

Randomness for positive dimension

- Frostman's Lemma: Let $\mathcal{A} \subseteq 2^{\omega}$ be compact. Then $\mathcal{H}^s(\mathcal{A}) > 0$ if and only if there exists a Radon probability measure μ with compact support contained in $\mathcal{A}$ such that

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(1) α is μ-random,

(2) μ is Δ^0_2 (relative to α).

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(1) For all n, $K(\alpha \upharpoonright_n) \geq h(n)$.

(2) α does not Turing-compute a ML-random sequence.

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Extracting randomness

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- Started with Von Neumann's idea how to turn a biased coin into an unbiased one.