Random Functions

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- Justification: randomness is base invariant. The *g*-ary expansion of a real number is ML-random iff the *h*-ary is (g, h ≥ 2).
- This invariance constrasts normality, which is not base-independent. (Cassels, 1959).
- Base-invariance of randomness has been proved by a number of people: Calude and Jürgensen (1994), Staiger (1998), Hertling and Weihrauch (1998).

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• Theorem: [Levin] If Φ is computable and ξ is μ -random, then $\Phi\xi$ is μ_{Φ} -random.

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- Given α , set $\alpha_0 = \alpha$ and let, for $n \ge 0$,

$$a_n = [\alpha_n]$$
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(Stop if $\{\alpha_n\} = 0$.)

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• The process is finite iff α is rational.

The continued fraction expansion

For irrational numbers, the partial convergents

$$[a_0, a_1, \dots a_n] := \frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots \frac{1}{a_n}}}} \qquad a_i \in \mathbb{N}$$

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• The continued fraction expansion induces a bijection between the irrational reals and the set of all infinite sequences of natural numbers, the Baire space $\mathbb{N}^{\mathbb{N}}$.

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$$\pi \mod 1 = [7, 15, 1, 292, 1, 1, \dots].$$

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- One can think of two possibilities to define this:
 - Say a cf is random if the sequence obtained by the binary expansion of the accordant real is random.

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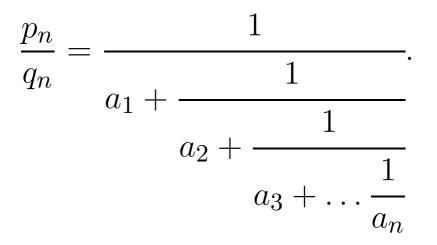
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- Key ingredient to the proof: Coding Theorem [Zvonkin-Levin].

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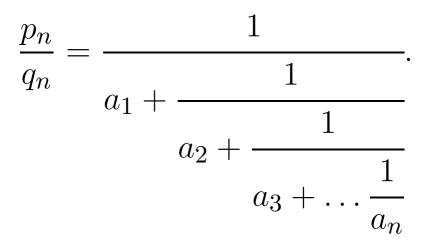
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• The diameter of a cylinder (as a subset of [0, 1]) is:

diam
$$[a_0, \dots a_n] = \frac{1}{q_n(q_n + q_{n-1})}$$

Effective measure

▶ A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is effectively null, if there is a recursive sequence (C_n) of r.e. sets such that for each n,

$$\mathcal{A} \subseteq \bigcup_{w \in C_n} [w]$$
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• A cf α is random if $\{\alpha\} \subseteq \mathbb{N}^{\mathbb{N}}$ is not effectively null.

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- Problem: The continued fraction expansion might code things more efficiently.

Comparing expansions

■ Let θ be irrational. Suppose $E_2(\theta) \upharpoonright_n = \theta_1 \theta_2 \dots \theta_n$ are the first *n* digits of its binary expansion. Set

$$r_n = r_n(\theta) = \sum_{i=1}^n \theta_i 2^{-i}$$
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• r_n and s_n are rational, hence their continued fraction expansions finite. Assume

$$CF(r_n) = [a_1, a_2, \dots, a_k]$$
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• Let $N = \max\{j : a_j = b_j\}$ and set $\pi_n(\theta) = [a_1, ..., a_N]$.

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The continued fraction algorithm yields

$$CF(r_5) = [3, 1, 5, 1, 1, 4, 2, 5, 1, 3]$$

$$CF(s_5) = [3, 1, 5, 1, 1, 5, 5, 1, 2, 1, 4, 3].$$

Therefore $\pi_5(\theta) = [3, 1, 5, 1, 1].$

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- Theorem: [Khintchine] For almost every $\alpha \in \mathbb{R}$,

$$\frac{1}{n}\log q_n(\alpha) \longrightarrow \frac{\pi^2}{12\log 2}$$

▲ Let (X, μ) be some measure space (Borel), $T: X \to X$ µ-preserving, that is, $\mu A = \mu T^{-1}A$ for all A Borel. T is ergodic if TA = A implies $\mu A \in \{0, 1\}$.

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- Theorem: [Birkhoff] For any continuous $f: X \to \mathbb{R}$, ergodic T, and μ -almost every $x \in X$, it holds that

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- Gauss map $x \mapsto \frac{1}{x} \mod 1$ describes the shift map for continued fractions. Its entropy is $\pi^2/6 \log 2$.
- However, the ergodic theorem is not effective (Vyugin, 1998).

Proving Invariance

• Theorem: The set of irrational numbers θ that do not satisfy Lochs' theorem is an effective nullset.

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- Theorem: [Faivre] For all $\varepsilon > 0$, there exist positive constants c, δ (depending on ε) with $0 < \delta < 1$ such that

$$\lambda\left(\left\{\xi \in \mathbb{R} : \left|\frac{|\pi_n \theta|}{n} - L\right| \ge \varepsilon\right\}\right) \le c\delta^n$$

for all integers $n \ge 1$ and with $L = 6 \log^2 2/\pi^2$.

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Result is based on transfer operators (Mayer, Ruelle).

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A sequence of infinitely many such fractions is given by the partial convergents of the continued fraction expansion of α.

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Examples:

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- **Proof** Roth's Theorem: For any algebraic α , for any $\varepsilon > 0$,

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^{2+\varepsilon}} \tag{-1}$$

has only finitely many solutions.

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Theorem: [Staiger] No Liouville number is a random real.

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This is an effective version of a theorem by Khintchine.