Measures and Their Random Reals

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Overview

The Dimension Problem

Consider any real X of positive effective Hausdorff dimension (positive asymptotic entropy), i.e.

$$\liminf_{n\to\infty} \frac{K(X\upharpoonright_n)}{n} > 0$$

Question

Does X compute a Martin-Löf random (λ-random) real?

The Von-Neumann Trick

Given a coin with bias p, use it to produce an unbiased coin toss:

Toss the coin twice. If the outcomes are identical, discard and toss anew, otherwise interpret HT as H and TH as T.

This is an example for extracting randomness from a random source of lower entropy. In the following we make this precise.

Notation for Cantor Space

- Elements of 2^ω are called reals.
- Finite initial segments of reals are strings.
- The (partial) prefix order on $2^{<\omega} \cup 2^\omega$ is denoted by $\sqsubset$.
- The basic clopen cylinder induced by a string σ is

$$[\![\sigma]\!] := \{X \in 2^\omega : \sigma \sqsubset X\}.$$

- If $C \subseteq 2^{<\omega}$, we write $[\![C]\!]$ to denote the open set

$$[\![C]\!] = \bigcup_{\sigma \in C} [\![\sigma]\!].$$

Measures on Cantor Space

Using Caratheodory's extension theorem, measures on 2^ω have a convenient representation.

It allows to identify premeasures with measures.

Definition

A measure on 2^ω is a function $\mu : 2^{<\omega} \to [0, \infty)$ such that for all $\sigma \in 2^{<\omega}$

$$\mu(\sigma) = \mu(\sigma 0) + \mu(\sigma 1).$$

If $\mu(2^\omega) = \mu(\epsilon) = 1$, μ is a probability measure.

In the following, all measures are assumed to be probability measures.

λ denotes the uniform measure $\lambda(\sigma) = 2^{-|\sigma|}$, which is isomorphic to Lebesgue measure on $[0, 1]$.

Randomness

Definition

Let μ be a computable measure.

1. A μ-Martin-Löf test is a uniformly enumerable sequence $(C_n)_{n\in\mathbb{N}}$, $C_n \subseteq 2^{<\omega}$ such that for every $n \in \mathbb{N}$

$$\sum_{\sigma \in C_n} \mu(\sigma) \leqslant 2^{-n}.$$

2. A sequence $R \in 2^{\omega}$ is Martin-Löf random with respect to μ (μ-random), if for all μ-Martin-Löf tests $(C_n)_{n\in\mathbb{N}}$

$$R \notin \bigcap_{n\in\mathbb{N}} [\![C_n]\!].$$

Bernoulli Randomness

If μ_p is the $(p, 1-p)$-Bernoulli measure ($0 < p < 1$), then for every μ_p-random real X it holds that

$$\liminf_{n\to\infty} \frac{K(X\restriction_n)}{n} = H(\mu_p) = -[p\log p + (1-p)\log(1-p)] > 0$$

(Levin, Lutz)

Von Neumann's trick ensures that, for reals random relative to a (computable) Bernoulli measure, it is always possible to extract a λ-random sequence.

Image Measures

Let μ be a measure and $f : 2^\omega \to 2^\omega$ be a continuous (Borel) function.

Define a new measure μ_f by setting

$$\mu_f(\sigma) = \mu(f^{-1}[\![\sigma]\!])$$

Observation: If $\mu\{X\} = 0$ for all $X \in 2^\omega$, then, for

$$F(X) = \mu\{Y : Y < X\},$$

it holds that $\mu_F = \lambda$.

The Isomorphism Theorem

Definition

Let μ be a measure on 2^ω.

- X is called an atom for μ if $\mu\{X\} > 0$.
- μ is continuous if it has no atoms.

Theorem

For every continuous measure μ there is a Borel isomorphism f of 2^ω such that $\mu_f = \lambda$.

Randomness Conservation

Idea: If the transformation f is computable, then it should preserve randomness, i.e. it should map a μ-random real to a μ_f-random one.

Note: If μ is a computable measure, then an atom of μ is μ-random iff it is computable.

Theorem (Levin, Kautz)

If a real is nonrecursive and random with respect to a computable measure, then it is Turing equivalent to a λ-random real.

Dimension and Randomness

So, is every real of positive dimension random with respect to a computable measure?

Theorem

For every rational $0 < r < 1$ there is a real X such that

$$\liminf_{n\to\infty} \frac{K(X\upharpoonright_n)}{n} = r,$$

and X is not random with respect to any computable measure.

Proof.

Join a 1-generic and a λ-random real with appropriate density. □

Randomness for Arbitrary Measures

But von Neumann's trick works for non-computable measures, too.

We extend Martin-Löf's notion of randomness to arbitrary measures.

Denote by $\mathcal{M}(2^\omega)$ the space of all probability measures on 2^ω.

Fact

$\mathcal{M}(2^\omega)$ is compact metrizable and Polish.

There is an effective representation of $\mathcal{M}(2^\omega)$ as a recursively bounded tree.

Randomness for Arbitrary Measures

Definition

Let $\mu \in \mathcal{M}(2^\omega)$. A Martin-Löf test for μ is a sequence $(V_n)_{n\in\mathbb{N}}$ of subsets of $2^{<\omega}$ such that (V_n) is uniformly r.e. in μ and for each n,

$$\sum_{\sigma \in V_n} \mu(\sigma) \leqslant 2^{-n}.$$

Note that every real is trivially random with respect to some μ if it is an atom of μ.

We are interested in the case when a real is non-trivially random.

The Duality Question

Is there a relation between the complexity of a real and the complexity of a measure that makes it (non-trivially) random?

Randomness is Non-Recursiveness

Recursive reals are only trivially random.

But they are the only reals for which this holds.

Theorem (Reimann and Slaman)

If X is a non-recursive real, then there is a measure μ such that $\mu\{X\} = 0$ and X is μ-random.

Randomness is Non-Recursiveness

Theorem (Reimann and Slaman)

If X is a non-recursive real, then there is a measure μ such that $\mu\{X\} = 0$ and X is μ-random.

The proof is based on the following idea:

- Image measures “push” measure forward. Randomness is conserved. The distribution function pushes forward to Lebesgue measure.
- We have a real and have to find a measure that makes it random.
- Instead of pushing measure forward, use a λ-random real and “pull” Lebesgue measure back.

Randomness is Non-Recursiveness

The basic outline of the proof is as follows. We want to make a real $X >_T \emptyset$ random relative to a measure μ such that $\mu\{X\} = 0$.

1. Kučera's Theorem yields that every real $\geqslant_T \emptyset'$ is Turing-equivalent to some λ-random real.
2. The theorem relativizes.
3. Use the Posner-Robinson Theorem to make X look like a jump (relative to some C).
4. Hence X is T-equivalent (relative to C) to some λ^C-random real R.
5. Use the T-equivalence to devise a Π^0_1 set of measures which are "pull backs" of λ.
6. Prove a basis theorem which yields that for one of these measures μ, R is $\lambda^{C \oplus \mu}$-random.
7. Conclude that X is μ^C-random and hence μ-random.

Relativizing Kučera's Theorem

Theorem

If $X \geqslant_{T(C)} C'$, then there exists a λ^C-random real R such that $X \equiv_{T(C)} R$.

Essential for the proof is the (relativized) Coding Lemma.

Lemma

Let (U_n) be a universal ML^C-test for some measure μ. Let $\mathcal{P}_n$ be the $\Pi^0_1(C \oplus \mu)$-class of μ^C-random reals given by the nth level of the test. Then there exists a function $g : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}^{>0}$ recursive in $C \oplus \mu$ such that for any $\sigma \in 2^{<\omega}$ and any $n \in \mathbb{N}$ it holds that

$$\mathcal{P}_n \cap [\![\sigma]\!] \neq \emptyset \quad \Rightarrow \quad \mu[\![\sigma]\!] \geqslant g(\sigma, n).$$

The Posner-Robinson Theorem

Theorem (Posner and Robinson)

If X is not recursive, then there exists a C such that

$$X \oplus C \geqslant_T C'$$

Pulling Back the Measure

Combining the relativized version of Kučera's Theorem with the Posner-Robinson Theorem, we conclude that there exists reals C, R such that

$$R \text{ is } \lambda^C\text{-random} \quad \text{and} \quad X \equiv_{T(C)} R$$

So assume that Φ and Ψ are Turing functionals recursive in C such that

$$\Psi(X) = R \quad \text{and} \quad \Phi(R) = X.$$

We will use Φ and Ψ to control the pull back of λ.

Pulling Back the Measure

We want to define $\mu(\sigma)$, $\sigma \in 2^{<\omega}$.

Let $\mathrm{Pre}(\sigma) := \{\tau \in 2^{\omega} : \ \Phi(\tau) \sqsupseteq \sigma \ \ \& \ \ \Psi(\sigma) \sqsubseteq \tau\}$.

If we want to define a measure μ with respect to which X is random, we have to satisfy two requirements:

1. The measure μ will dominate an image measure induced by Φ. This will ensure that any Martin-Löf random sequence is mapped by Φ to a μ-random sequence.
2. The measure must not be atomic on X.

To meet these requirements, we restrict the values of μ in the following way (if Φ and Ψ are defined):

$$\lambda(\mathrm{Pre}(\sigma)) \leqslant \mu(\sigma) \leqslant \lambda(\Psi(\sigma))$$

.

Pulling Back the Measure

(1) $\mathrm{Pre}(\sigma) := \{\tau \in 2^\omega : \Phi(\tau) \sqsupseteq \sigma \;\; \& \;\; \Psi(\sigma) \sqsubseteq \tau\}$.

(2) $\lambda(\mathrm{Pre}(\sigma)) \leqslant \mu(\sigma) \leqslant \lambda(\Psi(\sigma))$

Every configuration given by (2) corresponds to a subtree of $\mathcal{M}(2^\omega)$.

Every string being enumerated into $\mathrm{Pre}(\sigma)$ cuts off a branch in $\mathcal{M}(2^\omega)$.

It follows that

$$M := \{\mu : \mu \text{ satisfies (2)}\}$$

is a Π^0_1 subset of $\mathcal{M}(2^\omega)$.

A Basis Theorem for Relative Randomness

We want to show that for some $\mu \in M$, X is μ-random.

Note that if (V_n) were a μ-ML^C-test covering X, then $\Phi^{-1}(V_n)$ would be a λ-ML$^{C \oplus \mu}$-test covering R.

So, what we need to show is that R is $\lambda^{C \oplus \mu}$-random for some $\mu \in M$.

Theorem (independently by Downey, Hirschfeldt, Miller, and Nies)

If $\mathcal{B} \subseteq 2^\omega$ is nonempty and Π^0_1 in C, then, for every R which is λ^C-random there is $A \in \mathcal{B}$ such that R is $\lambda^{C \oplus A}$-random.

The proof is essentially a compactness argument.

Randomness and Continuous Measures

It seems desirable to further classify the reals by their randomness properties in the sense of the Duality Problem.

Can we improve the result that every non-recursive real is random with respect to some measure in the sense that it is actually random with respect to a measure without atoms, i.e. a continuous measure?

The answer turns out to be “no”. For some reals atoms are essential.

Not Continuously Random Reals

Call a real continuously random if it is random with respect to some continuous measure. We denote the class of reals that are not continuously random by NCR.

Theorem (Kjos-Hanssen and Montalban)

If X is a member of a countable Π^0_1 class, then $X \in \mathrm{NCR}$.

Proof.

- With respect to a continuous measure, every countable set has always measure 0.
- A Π^0_1 class of measure 0 cannot contain a random real.

□

Facts About NCR

- Members of countable Π^0_1 classes are hyperarithmetic.
  - Kreisel, 1959
- There are members of countable Π^0_1 classes at all levels of the hyperarithmetical hierarchy.
  - Cenzer, Clote, Smith, Soare, and Wainer, 1986
- What about elements of NCR outside the hyperarithmetic reals?
- Is NCR countable?
- NCR is a Π^1_1 class without a perfect subset.
  - It follows from work by Mansfield, Solovay that NCR $\subseteq \mathbb{L}$.

A Sufficient Condition for Continuous Randomness

What does ensure the existence of a continuous measure that makes a real random?

An analysis of the proof of the "Every Nonrecursive Real is Random"-Theorem yields the following condition:

If $X \leqslant_T R$, where R is a λ-random real, and $R \leqslant_{wtt} X$, then X is random with respect to some continuous measure.

Woodin's Theorem and NCR

Woodin showed if X is not hyperarithmetic, then there exists some C such that

$$X \oplus C \geqslant_{tt(C)} C'.$$

Woodin's theorem enables us, when outside the hyperarithmetical sets, to code a real into a tree of random reals (relative to some C) by means of a wtt-reduction.

Theorem

If X is not hyperarithmetic, then there exists a continuous measure μ such that X is μ-random.

Questions and Problems

- Is every real in NCR a member of a countable Π^0_1 class, i.e. is NCR the union of all countable Π^0_1 classes?
- What is the class of reals which are 2-random with respect to some (continuous) measure? Which reals are never (continuously) 2-random?
  - Working conjecture (Slaman): There are only countably many not continuously 2-random reals.
- Which reals compute a measure with respect to which they are random?
  - Note that such reals can, by relativizing Levin-Kautz, compute a λ-random real.
  - Working conjecture: There exists an upper cone of such reals.

Questions and Problems

- What is the complexity needed for a measure that makes a complex real random?
  - Working conjecture: If

$$\liminf_n \frac{K(X\lceil_n)}{n} = s > 0,$$

then there exists a Δ^0_2-computable measure μ such that

$$\mu(\sigma) \leqslant c2^{-|\sigma|s},$$

with respect to which X is random.