Kolmogorov-Loveland Randomness and Stochasticity

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March 19, 2005

Overview

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Schnorr's Criticism

- Martin-Löf randomness is a widely accepted formulation of randomness for individual sequences.
- ML-randomness coincides with incompressibility in terms of (prefix-free) Kolmogorov complexity.
- Schnorr's criticism: ML-randomness is algorithmically too strict. Notions of randomness should be based on computable objects, e.g. computable betting games.
- This talk: Is it possible to do this and still obtain a randomness concept as powerful as ML-randomness/incompressibility?
- Key ingredient: non-monotonicity.

Betting Games

Given: unknown infinite binary sequence A

A round in the game

- Start with a capital of 1.
- Select a position $k \in \mathbb{N}$ and specify a stake $v \in [0, 1]$.
- Predict the bit $A(k)$.
- If the prediction is correct, the capital is multiplied by $1 + v$. Otherwise the stake is lost.
- Continue: pick a new position not selected before.

A	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	…

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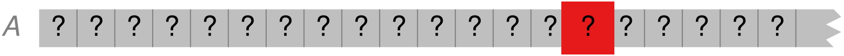

Capital: 1   Stake: 0.5

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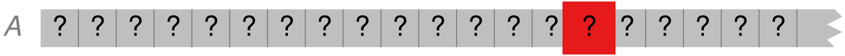

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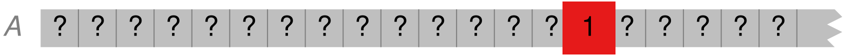

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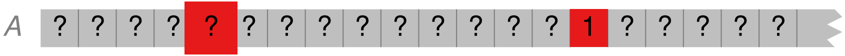

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Betting Strategies

Definition

- A betting strategy is a partial function that, given the outcomes of the previous rounds, determines
  - the position on which to bet next,
  - the stake to bet,
  - the value predicted.
- Formally, a betting strategy is a partial mapping

$$b : (\mathbb{N} \times \{0, 1\})^* \to \mathbb{N} \times [0, 1] \times \{0, 1\}.$$

- A betting strategy is monotone if the positions to bet on are chosen in an increasing order.
- A betting strategy b succeeds on sequence S if the capital grows unbounded when playing against S according to b.

Randomness as Unpredictability

Definition

- A sequence is Kolmogorov-Loveland random (KL-random) if no (partial) computable betting strategy succeeds on it.
- A sequence is computably random if no computable monotone betting strategy succeeds on it.

Randomness as Typicalness

Definition

- A Martin-Löf test (ML-test) is a uniformly computable sequence $(V_n)_{n\in\mathbb{N}}$ of c.e. sets of strings such that for all n,

$$\sum_{\sigma\in V_n} 2^{-|\sigma|} \leqslant 2^{-n}.$$

- An ML-test (V_n) covers a sequence A if $(\forall n)(\exists\sigma \in V_n)\ \sigma \sqsubset A$.
- A sequence is Martin-Löf random (ML-random) if it is not covered by an ML-test.
- A Schnorr test is an ML-test (V_n) such that the real number $\sum_{\sigma\in V_n} 2^{-|\sigma|}$ is uniformly computable. A sequence is Schnorr random if it not covered by a Schnorr test.

Relations between Randomness Notions

ML-random

$\Downarrow$

KL-random

$\Downarrow$

computably random

$\Downarrow$

Schnorr random

Relations between Randomness Notions

Open Question (Muchnik, Semenov, and Uspensky, 1998)

Is KL-randomness equivalent to ML-randomness?

ML-random

$\Updownarrow$?

KL-random

$\Downarrow \not\Uparrow$

computably random

$\Downarrow \not\Uparrow$

Schnorr random

Resource-Bounded Betting Games

Buhrman, van Melkebeek, Regan, Sivakumar, and Strauss (2000) studied resource-bounded betting strategies.

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Some Results

- If pseudorandom generators computable in exponential time (E, EXP) with exponential security exist, then every betting strategy computable in exponential time can be simulated by an exponential time monotone betting strategy.
- If exponential time betting strategies have the finite union property, then BPP $\neq$ EXP.

Example: Computably Enumerable Sets

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No partial computable non-monotonic betting strategy can succeed on all computably enumerable sets.

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No partial computable non-monotonic betting strategy can succeed on all computably enumerable sets.

Given a computable betting strategy b, define a c.e. set W such that b does not succeed on W by enumerating elements into W according to the places selected by b.

Example: Computably Enumerable Sets

Fact

There exist computable non-monotonic betting strategies b_0 and b_1 such that for every c.e. set W, at least one of b_0 and b_1 will succeed on W.

b_0 succeeds on all rather sparse sets, whereas b_1 succeeds if a lot of elements are enumerated into the set W.

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Failure of finite union property

Computable betting strategies do not have the finite union property.

Kolmogorov Complexity

Definition

Let U be a universal Turing machine. For a string σ define the Kolmogorov complexity C of a string as

$$\mathrm{C}(\sigma) = \mathrm{C}_U(\sigma) = \min\{|p| : p \in \{0,1\}^*, \; U(p) = \sigma\},$$

i.e. $\mathrm{C}(\sigma)$ is the length of the shortest U-program for σ.

Fact (Kolmogorov; Solomonoff)

C *is independent of the choice of* U*, up to an additive constant.*

Variant

Prefix-free complexity K. Based on prefix-free Turing-machines – no two converging inputs are prefixes of one another.

The Complexity of Martin-Löf Random Sequences

Theorem (Schnorr)

Given a sequence A, if there exists a function

$$h : \mathbb{N} \to \mathbb{N}$$

such that for all n,

$$\mathrm{K}(A \upharpoonright_{h(n)}) \leqslant h(n) - n,$$

then A is not ML-random.

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The Complexity of KL-Random Sequences

Theorem (Muchnik)

Given a sequence A, if there exists a computable function

$$h : \mathbb{N} \to \mathbb{N}$$

such that for all n,

$$\mathrm{K}(A \upharpoonright_{h(n)}) \leqslant h(n) - n,$$

then A is not KL-random.

The Complexity of KL-Random Sequences

Note that this is fails for computably random sequences. In fact, there can be computably random sequences of very low complexity. [Muchnik; Merkle]

Extracting Subsequences

Let Z be an infinite, co-infinite subset of $\mathbb{N}$.

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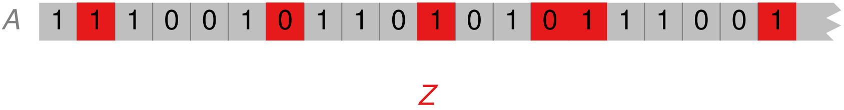

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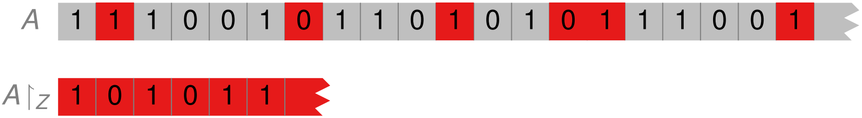

Generalized Joins

Definition

The Z-join of two sequences A_0, A_1,

$$A_0 \oplus_Z A_1,$$

is defined as the unique sequence A such that

$$A \upharpoonright_Z = A_1 \quad \text{and} \quad A \upharpoonright_{\overline{Z}} = A_0.$$

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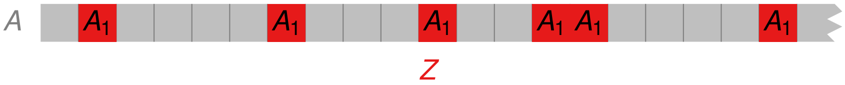

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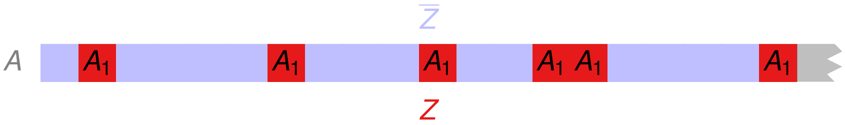

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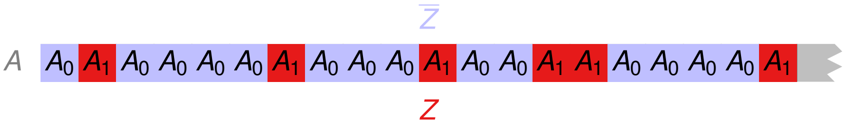

Splitting Properties

If we split a KL-random sequence effectively, both subsequences obtained must be KL-random relative to each other.

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Observation

Let Z be a computable, infinite and co-infinite set of natural numbers, and let $A = A_0 \oplus_Z A_1$. A is KL-random if and only if

$$A_0 \text{ is } \mathrm{KL}^{A_1}\text{-random} \quad \text{and} \quad A_1 \text{ is } \mathrm{KL}^{A_0}\text{-random.}$$

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Proof of "$\Rightarrow$":

- Suppose b_1 computable in A_1 succeeds on A_0.
- Devise betting strategy successful on A:
  - Scan the Z-positions of the sequence (corresponding to the places where A_1 is coded).
  - Find a new initial segment which allows to compute a new value of b_1.
  - Bet on the $\overline{Z}$-positions of the sequence according to b_1.

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- Suppose neither A_0 nor A_1 is ML-random.
- Then there are Martin-Löf tests $(U_n^0 \colon n \in \mathbb{N})$ and $(U_n^1 \colon n \in \mathbb{N})$ with $U_n^i = \{u_{n,0}^i, u_{n,1}^i, \ldots\}$, such that (U_n^i) covers A_i.
- Define functions f_0, f_1 by $f_i(n) = \min\{k \in \mathbb{N} \colon u_{n,k}^i \sqsubset A_i\}$.
- For some i, $(\overset{\infty}{\exists} m)\ f_i(m) \geqslant f_{1-i}(m)$.

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Theorem

Let Z be a computable, infinite and co-infinite set of natural numbers. If the sequence $A = A_0 \oplus_Z A_1$ is KL-random, then at least one of A_0, A_1 is Martin-Löf random.

- Define a new test (V_n) by

$$V_n = \bigcup_{m>n} \{u_0^{1-i}, \ldots, u_{f_i(m)}^{1-i}\}.$$

- (V_n) is a Schnorr test relative to the oracle A_i and covers A_{1-i}, so A_{1-i} is not Schnorr^{A_i}-random.
- KL-randomness implies Schnorr-randomness, so A_{1-i} is not KL^{A_i}-random, and hence A is not KL-random.

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This result can be improved.

Z has density δ if $\lim_{m\to\infty} |\{Z \cap \{0,\dots,m-1\}|/m = \delta$.

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Theorem

Let A be a KL-random sequence and let $\delta < 1$ be rational. Then there is a computable set Z of density at least δ such that $A\restriction_Z$ is ML-random.

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Proof uses a result by Van Lambalgen (1987), who showed that $A = A_0 \oplus_Z A_1$ is ML-random if and only if A_0 is ML-random and A_1 is ML^{A_0}-random.

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A Counterexample

The splitting property of KL-random sequences is not a sufficient criterion for ML-randomness.

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Theorem

There is a sequence A which is not computably random such that for each computable, infinite and co-infinite set Z, $A\restriction_Z$ is ML-random.

Kolmogorov-Loveland Stochasticity

One can modify betting strategies to obtain the concept of selection rules.

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Selection rules

- Select a position $k \in \mathbb{N}$.
- Specify whether to include the bit $A(k)$ in the selected subsequence.
- After the bit is revealed pick a new position not selected before.

Kolmogorov-Loveland Stochasticity

Definition

A sequence A is Kolmogorov-Loveland stochastic if every infinite subsequence of A selected by a computable selection rule has limit frequency $1/2$.

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Every ML-random sequence is KL-stochastic.

Shen (1988) showed that there are KL-stochastic sequences not ML-random.
He used Bernoulli distributions $(\beta_n, 1 - \beta_n)$, with

$$\sum_n (1/2 - \beta_n)^2 = \infty$$

.

Effective Dimension

- There exists an interesting connection between the asymptotic complexity of sequences and Hausdorff dimension.
- Hausdorff dimension is defined via Hausdorff measures. Similar to Lebesgue measure, one can define effective versions [Lutz 2000].
- Effective dimension, $\dim_H^1$, can be characterized in terms of Kolmogorov complexity.

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- Effective dimension, $\dim_{\mathrm{H}}^1$, can be characterized in terms of Kolmogorov complexity.

Theorem (Ryabko; Mayordomo)

The effective dimension of a sequence A is given by

$$\dim_{\mathrm{H}}^1 A = \liminf_{n\to\infty} \frac{\mathrm{K}(A\restriction_n)}{n}.$$

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If R is KL-stochastic, then $\dim_{\mathrm{H}}^{1} R = 1$.

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If R is KL-stochastic, then $\dim_H^1 R = 1$.

This implies, in particular, that all KL-random sequences have dimension 1, too.

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- The main ingredient of the proof:

$$A = A_0 \oplus_Z A_1 \text{ KL-stochastic } \Rightarrow \dim_{\mathrm{H}}^{1} A_0 = 1 \text{ or } \dim_{\mathrm{H}}^{1} A_1 = 1.$$

- Then relativize to obtain arbitrary dense subsequences of dimension 1.
- Finally, use

$$\dim_{\mathrm{H}}^{1} A_0 \oplus_Z A_1 \geqslant \delta_Z \dim_{\mathrm{H}}^{1} A_1 + (1 - \delta_Z) \dim_{\mathrm{H}}^{1, A_1} A_0,$$

where δ_Z denotes the density of Z. (Proof uses symmetry of algorithmic information.)

The Dimension of KL-Stochastic Sequences

$A = A_0 \oplus_Z A_1$ KL-stochastic $\Rightarrow$ $\dim^1_H A_0 = 1$ or $\dim^1_H A_1 = 1$.

The Dimension of KL-Stochastic Sequences

Lemma

For every computable, infinite, co-infinite set Z,

$$A = A_0 \oplus_Z A_1 \text{ \textit{KL-stocjastic}} \Rightarrow \dim^1_{\mathrm{H}} A_0 = 1 \text{ \textit{or} } \dim^1_{\mathrm{H}} A_1 = 1.$$

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- Suppose $\dim^1_{\mathrm{H}} A_0, A_1 < \alpha < 1$.
- Find a selection rule that selects from A an unbalanced subsequence.
- The set of α-compressible strings ($\mathrm{K}(w) \leqslant \alpha|w|$) can be effectively enumerated $\{w_0, w_1, \dots\}$.
- If a sequence has infinitely many α-compressible initial segments, it must also have infinitely many α-compressible substrings.

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- Idea: Use the compressible substrings to select an unbalanced subsequence.
- Known techniques: From a finite set of compressible strings we can compute a selection rule which does this. (Conversion of martingales into selection rules)
- The function

  $$g_r(m) = \mu i \left[w_i \text{ substring of } A_r \text{ at position } m \text{ and } |w_i| \geqslant cm \right]$$

  is total (for any constant c).

Lemma

For every computable, infinite, co-infinite set Z,

$$A = A_0 \oplus_Z A_1 \text{ KL-stocjastic } \Rightarrow \dim_{\mathrm{H}}^1 A_0 = 1 \text{ or } \dim_{\mathrm{H}}^1 A_1 = 1.$$

- Let

$$m_{t+1} = m_t + \max\{|w_i| : i \leqslant \max\{g_0(m_t), g_1(m_t)\}\}.$$

  The m_t will be the "selection blocks".

- Now w.l.o.g. assume that

$$\exists^\infty t \, (g_0(m_t) \leqslant g_1(m_t)).$$

  We will use A_1 for scanning, A_0 for selecting.

The Dimension of KL-Stochastic Sequences

Lemma

For every computable, infinite, co-infinite set Z,

$$A = A_0 \oplus_Z A_1 \textit{ KL-stocjastic } \Rightarrow \dim^1_{\mathrm{H}} A_0 = 1 \textit{ or } \dim^1_{\mathrm{H}} A_1 = 1.$$

- By scanning only bits of A_1, we can, for every t, compute $g_1(m_t)$. From the strings $\{w_0, \dots w_{g_1(m_t)}\}$ compute a selection rule as described above.
- Infinitely often some $w \in \{w_0, \dots w_{g_1(m_t)}\}$ is a substring of A_0 at m_t, so the selection rule selects a long, unbalanced substring from $w = A_0(m_t) \dots A_0(m_t + |w| - 1)$.

Conclusion

- Non-monotonicity makes (computable) betting strategies much more powerful.
- In many ways, KL-randomness behaves like Martin-Löf randomness.
- However, none of the properties studied is a sufficient condition for ML-randomness; on the contrary, there are examples “far from ML-randomness”.
- A proof that KL-randomness is equivalent to ML-randomness would would give a striking argument against Schnorr’s criticism of Martin-Löf randomness.