The Metamathematics of Algorithmic Randomness

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The Initial Question

Question

For which reals $X \in 2^\omega$ does there exist (a representation of) a measure μ such that X is random for μ?

Representation of Measures

We want to generalize Martin-Löf randomness to arbitrary measures. For this, we have to access measures as oracles.

- In Cantor space we can simply code the rational approximations to a measure in a real.
- More general, if a space X is Polish, so is the space $\mathcal{M}(X)$ of all probability measures on X (under the weak topology). Also, if X is compact metrizable, so is $\mathcal{M}(X)$.

Note that there are various ways to represent a measure: Cauchy sequences, list of basic open balls it is contained in, etc.

- There might not be a least representation in terms of Turing-degree.

Martin-Löf tests

Definition

Let m be a representation of some $\mu \in \mathcal{M}(2^\omega)$, and let $n \geqslant 1$.

- An n-Martin-Löf test for m is a sequence $(V_n)_{n\in\mathbb{N}}$ of subsets of $2^{<\omega}$ such that (V_n) is uniformly r.e. in $m^{(n-1)}$ and for each n,
$$\sum_{\sigma\in V_n} \mu(N_\sigma) \leqslant 2^{-n}.$$
- A real X is n-random for m if for every n-Martin-Löf test for m,
$$X \notin \bigcap_k \bigcup_{\sigma\in V_k} N_\sigma$$

Non-Trivial Randomness

Note that every real is trivially random with respect to some μ if it is an atom of μ.

- We are interested in the case when a real is non-trivially random.

Theorem (Reimann and Slaman)

For any real X, there exists (a representation of) a measure μ such that $\mu(\{X\}) \neq 0$ and X is 1-random for μ if and only if X is not recursive.

In the proof there is no control over the measure obtained.

- Atoms cannot be avoided.
- Uses a special (though natural) representation of $\mathcal{M}(2^\omega)$ as a particular Π^0_1 class.

Non-Trivial Randomness

Features needed in the proof:

- Conservation of randomness:
  - If $f : 2^\omega \to 2^\omega$ is continuous, μ a measure, then $\mu_f(A) := \mu(f^{-1}(A))$ defines the image measure.
  - If f is effective and X is random for μ, $f(X)$ is random for μ_f.
- Randomness of cones:
  - Kucera's coding argument shows that every degree above $\emptyset'$ is random.
  - Relativize this using the Posner-Robinson Theorem.

Neutral Measure

A similar result can be obtained by using a neutral measure, relative to which every real looks random.

Theorem (Levin; Gacs)

There exists a measure μ *such that for every* X, $t_\mu(X) \leqslant 1$, *where* $t_\mu(X)$ *is a universal test for randomness for* μ.

- The proof uses the combinatorial Sperner Lemma.
- Works only for compact spaces.

Continuous Randomness

In the following, we will concentrate on continuous, i.e. non-atomic measures.

- For these, the transformation of measures and randomness (and with it the representation of the measure) is particularly well-behaved.
- Classical result: For every continuous measure μ there is a Borel isomorphism f of 2^ω such that $\mu = \lambda_f$, λ being Lebesgue measure.

Continuous Randomness

An effective version

Theorem (Levin; Kautz; Reimann and Slaman)

Let X be a real. The following are eqivalent.

(i) *X is truth-table equivalent to a Martin-Löf random real.*

(ii) *X is random for a continuous recursive measure.*

(iii) *X is random for a continuous dyadic recursive measure.*

(iv) *There exists a recursive functional Φ which is an order-preserving homeomorphism of 2^ω such that $\Phi(X)$ is Martin-Löf random.*

Hence we can define (continuous) randomness degree-theoretically.

The Class NCR

Question

Which level of logical complexity guarantees continuous randomness?

Let NCR_n be the set of all reals which are not n-random relative to any continuous measure.

- Kjos-Hanssen and Montalban: Every member of a countable Π^0_1 class is contained in NCR_1. (It follows that elements of NCR_1 can be found at arbitrary high levels of the hyperarithmetical hierarchy.)
- Reimann and Slaman: $\mathrm{NCR}_1 \subseteq \Delta^1_1$.

The proofs are arguments tailored for $n = 1$ and do not carry over to higher levels of randomness.

Theorem

Kleene's $\mathcal{O}$ is an element of NCR_3.

Based on this, one can use the theory of jump operators (Jockusch ans Shore) to obtain a whole class of examples.

Proof:

- Tree representation of $\mathcal{O}$:

$$\mathcal{O} = \{e : \text{ the } e\text{th recursive tree } T_e \subseteq \omega^{<\omega} \text{ is well-founded}\}.$$

- Suppose $\mathcal{O}$ is 3-random for some μ.
- We want to use domination properties of random reals.

The Class NCR

Examples of higher order

- Well-known (Kurtz and others): If X is n-random for μ, $n > 1$, then every function $f \leqslant_T X$ is dominated by a function recursive in μ'.
- Therefore, μ' computes a uniform family $\{g_e\}$ of functions dominating the leftmost infinite path of T_e.
- Use compactness to infer: For every e, the following are equivalent.
  (i) T_e is well-founded.
  (ii) The subtree of T_e to the left of g_e is finite.
- The latter condition is $\Pi^0_1(\mu')$, hence $\mathcal{O}$ is $\Pi^0_2(\mu)$.
- But this is impossible if $\mathcal{O}$ is 3-random for μ.

Lower Bounds for Continuous Randomness

In general, can we give a distinct bound on NCR_n like in the case $n = 1$?

- There is some evidence that NCR_n grows very quickly with n.
- Can we give an upper bound?

Theorem (Slaman)

For all n, NCR_n is countable.

NCR_n is Countable

Proof:

- Show that the complement of NCR_n contains an upper Turing cone.
  - Show that the complement of NCR_n contains a Turing invariant and cofinal Borel set. We can use the set of all Y that are Turing equivalent to some $Z \oplus R$, where R is $(n+1)$-random relative to Z.
  - Use Martin's result on Borel Turing sets to infer that the complement of NCR_n contains a cone.
- Go on to show that the elements of NCR_n are definable at a rather low level of the constructible universe.
  - $\mathrm{NCR}_n \subseteq L_{\beta_n}$, where β_n is the least ordinal such that

    $$L_{\beta_n} \models \mathrm{ZFC}^- + \text{there exist } n \text{ many iterates of the power set of } \omega,$$

    where ZFC^- is Zermelo-Fraenkel set theory without the Power Set Axiom.
  - Note that L_{β_n} is countable.

Is the Metamathematics Necessary?

Question

Do we need to use metamathematical methods to prove the countability of NCR_n?

We make fundamental use of Borel determinacy; this suggests to analyze the metamathematics in this context.

Friedman's Result on Borel Determinacy

The necessity of iterates of the power set is known from a famous result by Friedman.

- Martin's proof of Borel determinacy starts with a description of a Borel game and produces a winning strategy for one of the players.
- The more complicated the game is in the Borel hierarchy, the more iterates of the power set of the continuum are used in producing the strategy.

Theorem (Friedman)

$\mathrm{ZFC}^- \nvdash$ *All $\boldsymbol{\Sigma}^0_5$-games on countable trees are determined.*

Martin later improved this to $\boldsymbol{\Sigma}^0_4$.

Friedman's Result on Borel Determinacy

Inductively one can infer from Friedman's result that in order to prove full Borel determinacy, a result about sets of reals, one needs infinitely many iterates of the power set of the continuum.

- The proof works by showing that there is a model of ZFC^- for which $\mathbf{\Sigma}^0_4$-determinacy does not hold.
- This model is L_{β_0} .

NCR and Iterates of the Power Set

We can work along similar lines to obtain a similar result concerning the countability of NCR_n.

Theorem

For every k, the statement

$$\text{For every } n,\ \mathrm{NCR}_n \text{ is countable.}$$

cannot be proven in

$$\mathrm{ZFC}^- + \text{there exists } k \text{ many iterates of the power set of } \omega.$$

The proof (for $k = 0$) shows that there is an n such that NCR_n is cofinal in the Turing degrees of L_{β_0}. Hence, NCR_n is not countable in L_{β_0}.