The Effective Dimension of Cones and Degrees

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Overview

Outer Measures
The Caratheodory Method

- A premeasure is a function $\rho : 2^{<\omega} \to \mathbb{R}_0^+ \cup \{\infty\}$
- One can obtain an outer measure μ_ρ from ρ by letting

$$\mu_\rho(\mathcal{X}) = \inf \left\{ \sum_i \rho(x_i) : \bigcup_i [x_i] \supseteq \mathcal{X} \right\}.$$

- $\mu = \mu_\rho$ is a countably subadditive, monotone set function. Restriction to sets $\mathcal{A}$ which satisfy

$$(\forall \mathcal{Y})\, \mu(\mathcal{Y}) = \mu(\mathcal{Y} \cap \mathcal{A}) + \mu(\mathcal{Y} \setminus \mathcal{A}),$$

  yields the measurable sets.
- The measurable sets form a σ-algebra, and μ is an additive set function on this σ-algebra.

Effectivizing Measures

— Let ρ be a computable premeasure, with μ_ρ the induced outer measure.

Definition A set $\mathcal{X} \subseteq 2^\omega$ has effectively μ_ρ-measure zero if there exists a uniformly computable sequence (C_n) of sets of strings such that for all n,

$$\mathcal{X} \subseteq \bigcup_{\sigma \in C_n} [\sigma] \quad \text{and} \quad \sum_{\sigma \in C_n} \rho(\sigma) \leqslant 2^{-n}.$$

Hausdorff Measures and Hausdorff Dimension

- Hausdorff measures $\mathcal{H}^s$ arise from the premeasures $\rho(\sigma) = 2^{-|\sigma|s}$, $s \geqslant 0$.
- It is obvious that $\mathcal{H}^s\mathcal{X} = 0$ implies $\mathcal{H}^t\mathcal{X} = 0$ for all $t > s$.

Definition The Hausdorff dimension of $\mathcal{X}$ is defined as

$$\dim_{\mathrm{H}} \mathcal{X} = \inf\{s \geqslant 0 : \mathcal{H}^s\mathcal{X} = 0\}.$$

Effective Hausdorff Dimension

Definition The effective Hausdorff dimension of $\mathcal{X}$ is defined as

$$\dim^1_{\mathrm{H}} \mathcal{X} = \inf\{s \in \mathbb{Q}^+_0 : \mathcal{X} \text{ is effectively } \mathcal{H}^s\text{-null}\}.$$

[Lutz 2000]

— There are single reals of non-zero dimension: every Martin-Löf random real has dimension one.

— Effective dimension has an important stability property:

$$\dim^1_{\mathrm{H}} \mathcal{X} = \sup\{\dim^1_{\mathrm{H}}\{A\} : A \in \mathcal{X}\}.$$

[Lutz 2000]

Effective Hausdorff Dimension

Effective dimension as degree of randomness

Theorem For every real A,

$$\dim_{\mathrm{H}}^{1} A = \liminf_{n\to\infty} \frac{K(A\restriction_n)}{n} =: \underline{\mathrm{K}}(A).$$

[Ryabko 1984; Mayordomo 2002]

— Given $0 < r < 1$ rational, let $Z_r = \{\lfloor n/r \rfloor : n \in \mathbb{N}\}$. Given a Martin-Löf random set X, define X_r by

$$X_r(m) = \begin{cases} X(n) & \text{if } m = \lfloor n/r \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\dim^1_{\mathrm{H}} X_r = r$.

— Let μ_p be a Bernoulli ("coin-toss") measure with bias $p \in \mathbb{Q} \cap [0,1]$, and let B be Martin-Löf random with respect to μ_p. Then

$$\dim^1_{\mathrm{H}} B = H(\mu_p) := -[p \log p + p \log(1-p)].$$

— Let U be a universal, prefix-free machine. Given a computable real number $0 < s \leqslant 1$, the binary expansion of the real number

$$\Omega^{(s)} = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-\frac{|\sigma|}{s}}$$

has effective dimension s [Tadaki 2002]. (Note that $\Omega^{(1)}$ is just Chaitin's Ω.)

Effective Hausdorff Dimension

The basic examples imply genuine random content

- Each of the three examples actually computes a Martin-Löf random real.
- This is obvious for the "diluted" sequence.
- For computable Bernoulli measures, one may use Von-Neumann's trick to turn a biased random real into a uniformly distributed random real. More generally, Levin (1970) and Kautz (1991) have shown that any real which is random with respect to computable measure computes a Martin-Löf random real.
- $\Omega^{(s)}$ computes a fixed-point free function. It is a left-computable real, and hence it follows from the Arslanov completeness criterion that $\Omega^{(s)}$ is Turing complete (and thus T-equivalent to a Martin-Löf random real).

The Dimension Problem

Are there "genuine" reals of non-integral dimension?

— The stability property implies that the Turing lower cone of each of the three examples has effective dimension 1.

Question Are there any Turing lower cones of non-integral dimension?

— This is an open problem. Any such lower cone would come from a real of non-integral dimension for which it is not possible to extract some content of higher degree of randomness effectively.

- For upper cones, the situation is quite clear.
- It is known that the Turing upper cone of a real has Lebesgue measure zero unless the real is computable [Sacks 1963].

Theorem For any real A, the many-one upper cone of A has (classical) Hausdorff dimension 1.

Lower Cones and Degrees

— The dimension of a lower cone and a degree coincide.

— This follows from the sparse coding technique: Given two reals $A \leqslant_r B$, choose a computable real R of density $\lim_n |R \cap \{0, \ldots, n-1\}|/n = 1$, and let C equal A on R and B on the complement of R.

— C will be r-equivalent to B and be of the same dimension as A. It follows that the dimension of the degree and the lower cone of a set coincide.

Many-One Reducibility

Theorem Let μ_p be a computable Bernoulli measure with bias p. If A is μ_p-random, then

$$B \leqslant_{\mathrm{m}} A \;\Rightarrow\; \dim^1_{\mathrm{H}} B \leqslant H(\mu_p).$$

[Reimann and Terwijn 2004]

— Proof. Given an m-reduction f, define $F = \{n : (\forall m < n) f(m) \neq f(n)\}$, so F is the set of all positions of B, where an instance of A is queried for the first time.

— F induces a Kolmogorov-Loveland place selection rule. If A is μ_p-random, this selection rule will yield a new sequence with the same limit frequency as A.

Weaker Reducibilities

- This technique does not extend to weaker reducibilities, since for Bernoulli measures the Levin-Kautz result holds for a total Turing reduction.
- Stephan (2005) was able to construct wtt-lower cone of non-integral effective dimension in a relativized world:

  There is a real A and an oracle B such that

  $$1/3 \leqslant \dim_{\mathrm{H}}^{B}\{D : D \leqslant_{\mathrm{wtt}}^{B} A\}) \leqslant 1/2.$$

Theorem For each rational α, $0 \leqslant \alpha \leqslant 1$, there is a real $A \leqslant_{\mathrm{wtt}} \emptyset'$ such that

$$\underline{K}(A) = \alpha \quad \text{and} \quad (\forall Z \leqslant_{\mathrm{wtt}} A)\, \underline{K}(Z) \leqslant \alpha.$$

- Requirements:

$$R_{\langle e,b\rangle} : Z = \Psi_e(A) \Rightarrow \exists (k \geqslant j)\, K(Z \upharpoonright_k) \leqslant^+ (\alpha + 2^{-b})k$$

  where (Ψ_e) is a uniform listing of wtt reduction procedures.

- We can assume each Ψ_e also has a certain (non-trivial) lower bound on the use g_e, because otherwise the reduction would decrease complexity anyway.

- We construct A inside the Π^0_1 class

$$\mathcal{P} = \{Z : (\forall n \geqslant n_0)\, K(Z \upharpoonright_n) \geqslant \lfloor \alpha n \rfloor\}$$

  (This ensures A has dimension at least α.)

- $\mathcal{P}$ is given as an effective approximation through clopen sets P_s.

- We approximate longer and longer initial segments σ_j of A, where σ_j is a string of length m_j, both σ_j, m_j controlled by R_j.

- Define a length k_j where we intend to compress Z, and let $m_j = g_e(k_j)$.
- Define σ_j of length m_j in a way that, if $x = \Psi_e^{\sigma_j}$ is defined then we compress it down to $(\alpha + 2^{-b_j})k_j$, by enumerating an appropriate request into a Kraft-Chaitin set L.
- The opponent's answer could be to remove σ_j from $\mathcal{P}$. (σ_j is not of high dimension.)
- In this case, the capital he spent for this removal exceeds what we spent for our request, so we can account our capital against his.
- Of course, usually σ_j is much longer than x. So we will only compress x when the measure of oracle strings computing it is large.

— We assume that $\mathcal{P}$ is effectively approximated by clopen sets P_s.

Lemma Let $\mathcal{C}$ be a clopen class such that $\mathcal{C} \subseteq P_s$ and $\mathcal{C} \cap P_t = \emptyset$ for stages $s < t$. Then

$$\Omega_t - \Omega_s \geqslant (\lambda\mathcal{C})^\alpha.$$

- In the course of the construction, some R_j might have to pick a new σ_j.
- In this case we have to initialize all R_n of lower priority ($n > j$).
- We have to make sure that this does not make us enumerate too much measure into L.
- We therefore have to assign a new length k_n to the strategies R_n.
- For this, it is important to know the use of the reduction related to R_j.

The Turing Case

— It remains an open problem whether there exists a Turing lower cone of non-integral effective dimension.

— This case appears to be much harder. It is, for instance, not even known whether there exists a set of non-integral dimension which does not compute a Martin-Löf random set.

Theorem There exists computable, non-decreasing, unbounded function f and a set A such that

$$K(A\restriction_n) \geqslant f(n)$$

and A does not compute a Martin-Löf random set.

[Kjos-Hanssen, Merkle, and Stephan 2004; Reimann and Slaman 2004,]