The Strength of the Besicovitch-Davies Theorem

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Abstract. A theorem of Besicovitch and Davies implies for Cantor space 2^ω that each $\boldsymbol{\Sigma}^1_1$ (analytic) class of positive Hausdorff dimension contains a $\boldsymbol{\Pi}^0_1$ (closed) subclass of positive dimension. We consider the weak (Muchnik) reducibility $\leq_w$ in connection with the mass problem $S(U)$ of computing a set $X \subseteq \omega$ such that the Σ^1_1 class U of positive dimension has a $\Pi^0_1(X)$ subclass of positive dimension.
We determine the difficulty of the mass problems $S(U)$ through the following results:
(1) Y is hyperarithmetic if and only if $\{Y\} \leq_w S(U)$ for some U;
(2) there is a U such that if Y is hyperarithmetic, then $\{Y\} \leq_w S(U)$;
(3) if Y is Π^1_1-complete then $S(U) \leq_w \{Y\}$ for all U.
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1 Introduction

One of the most useful properties of Lebesgue measure λ is its *regularity*: For any measurable set E,

$$\begin{aligned} \lambda(E) &= \sup\{\lambda(K) \colon K \subseteq E, K \text{ compact}\} && (1) \\ &= \inf\{\lambda(U) \colon U \supseteq E, U \text{ open}\}. && (2) \end{aligned}$$

This implies that (using the appropriate convergence theorems), for measure theoretic considerations, E can be replaced by a G_δ or F_σ set of the same measure, simplifying the complicated topological structure of arbitrary Borel sets.

The regularity properties (1) and (2) hold more generally for any positive Borel measure on a σ-compact Hausdorff space in which any compact set has finite measure, but fails to be true in general. It is one of the major results in geometric measure theory that the s-dimensional *Hausdorff measures* $\mathcal{H}^s$ are still *inner regular*[1].

[1] While *outer regularity* (2) fails for Hausdorff measures in general (open sets have infinite $\mathcal{H}^s$-measure for $s < 1$), one can still find, for any measurable set E, a G_δ set of the same Hausdorff measure. This is often referred to as *G_δ-regularity* (see Rogers[21]).

Theorem 1. *For any analytic ($\mathbf{\Sigma^1_1}$) set $E \subseteq \mathbb{R}^n$ and for any $s \geq 0$,*

$$\mathcal{H}^s(E) = \sup\{\mathcal{H}^s(K)\colon K \subseteq E, K \text{ compact}\}.$$

Theorem 1 was shown by Besicovitch [1] for $\mathbf{\Sigma^0_3}$ sets and extended by Davies [2] to $\mathbf{\Sigma^1_1}$ sets. It was subsequently generalized to various non-Euclidean settings. In 1995, Howroyd [10] showed that inner regularity holds for $\mathcal{H}^s$ on any compact metric space, in particular for *Cantor space* 2^ω with the standard metric

$$d(X,Y) = \begin{cases} 2^{-\min\{n\colon X(n)\neq Y(n)\}} & X \neq Y \\ 0 & X = Y. \end{cases}$$

In the following, we refer to the inner regularity property of Hausdorff measure in Euclidean and compact metric spaces simply as the *Besicovitch-Davies Theorem.*

The hierarchies of effective descriptive set theory allow for a further ramification of regularity properties. Any (boldface) Borel set is effectively (lightface) Borel relative to a parameter. Hence we can, for instance, given a (lightface) Σ^0_α set, measure how hard it is to find a $\Sigma^0_2(Y)$ subset of the same measure, by proving lower bounds on the parameter $Y \in 2^\omega$.

Dobrinen and Simpson [4] investigated this question for Σ^0_3 sets in Lebesgue measure and discovered an interesting connection with measure-theoretic *domination properties.* Subsequently, measure-theoretic domination properties were linked to LR-reducibility, a reducibility concept from algorithmic randomness. Recently, Simpson [24] gave a complete characterization of the regularity problem for Borel sets with respect to Lebesgue measure. One of his results states that the property that every $\Sigma^0_{\alpha+2}$ (α a recursive ordinal) subset of 2^ω has a $\Sigma^0_2(Y)$ subset of the same Lebesgue measure holds if and only if $0^{(\alpha)} \leq_{LR} Y$. His paper [24] also contains a survey of previous results along with an extensive bibliography.

In this paper, we study the complexity of the corresponding inner regularity for Hausdorff measure on 2^ω. We will see that, in contrast to the case of Lebesgue measure, finding subsets of positive Hausdorff measure can generally not be done with the help of a *hyperarithmetical* oracle. The core observation is that determining whether a set of reals has *positive Hausdorff measure* is more similar to determining whether it is *non-empty* than to determining whether it has *positive Lebesgue measure.*

Determining the exact strength of the Besicovitch-Davies Theorem is not only of intrinsic interest. A family of important problems in theoretical computer science ask some version of the question to what extent randomness (which is a useful computational tool) can be extracted from a weakly random source (which is often all that is available). Such questions can also be expressed in computability theory. The advantage, and simultaneously the disadvantage, of doing so is that one abstracts away from considering any particular model of efficient computation. One way to conceive of weak randomness is in terms of *effective Hausdorff dimension.* Miller [18] and Greenberg and Miller [8] obtained a negative result for randomness extraction: there is a real of effective Hausdorff dimension 1, that does not Turing compute any Martin-Löf random real. Despite this negative result, effective Hausdorff dimension, which is a "lightface" form of Hausdorff dimension, has independent interest, as it seems to offer a way to redevelop much of geometric measure theory (for example Frostman's Lemma [20]) in a more effective way.

Another conception of weak randomness comes from considering sets that differ from Martin-Löf random sets only on a sparse set of bits [13], or sets that are subsets of Martin-Löf sets [9, 12, 14]. Actually, these conceptions are related, as we will try to illustrate with the help of the set MIN of all reals that have minimal Turing degree.

The following result seems rather surprising.

Theorem 2. *The set* MIN *has Hausdorff dimension 1.*

Proof. This is merely a relativization of the theorem of Greenberg and Miller [8] that there is a minimal Turing degree of effective Hausdorff dimension 1. [2] □

Theorem 2 says that high effective Hausdorff dimension is not sufficient to be able to extract randomness. It can also be used to deduce that infinite subsets of random sets are not sufficiently close to being random, either.

The set MIN is Π^0_4, so by the Besicovitch-Davies Theorem, MIN has a closed subset C that still has Hausdorff dimension as close to 1 as desired. Each closed set C in Cantor space is $\Pi^0_1(X)$ for some oracle X. By a reasoning similar to [3, Theorem 4.3], each X-random closed set contains a member of C. It follows by reasoning as in [14] that each X-random set has an infinite subset of minimal Turing degree - in particular an infinite subset that Turing computes no 1-random (Martin-Löf random) set. Thus, if X could be chosen recursive, we would have a positive answer to the following question.

Question 1. Does each 1-random subset of ω have an infinite subset that computes no 1-random sets?

A partial answer to this question is known, using other methods:

Theorem 3 **([12]).** *Each 2-random set has an infinite subset that computes no 1-random sets.*

But it is easy to see that the set X just referred to cannot be chosen recursive. To wit, by the computably enumerable degree basis theorem there is no nonempty Π^0_1 class consisting entirely of sets of minimal Turing degree. In the present article we show that X can be taken recursive in Kleene's O, but in general, for arbitrary Σ^1_1 classes (or even just arbitrary Π^0_2 classes) in place of MIN, X cannot be taken hyperarithmetical.

[2] Their work in turn builds on the construction of a diagonally non-recursive function of minimal Turing degree in Kumabe and Lewis [16].

We expect the reader to be familiar with basic descriptive set theory and the effective part on hyperarithmetic sets and Kleene's $\mathcal{O}$. Standard references are [22] and [23]. We also assume basic knowledge of Hausdorff measures and dimension, as can be found in [21]. The proofs are rather succinct, but details can easily be filled in using basic results and methods of the above theories.

2 Index set complexity

Recall that the *Hausdorff dimension* of a set E in a metric space X is defined as

$$\dim_{\mathrm{H}}(E) = \inf\{s \geq 0 \colon \mathcal{H}^s(E) = 0\}.$$

To keep the presentation simple, we concentrate on the problem of finding, given a set $E \subseteq 2^\omega$, a closed subset of positive Hausdorff dimension. Note that if $\dim_{\mathrm{H}}(E) > 0$ and E is $\boldsymbol{\Sigma}^1_1$, then by the Besicovitch-Davies Theorem there exists a closed subset $C \subseteq E$ such that $\dim_{\mathrm{H}}(C) > 0$.

Davies' argument is based on the representation of analytic sets via *Souslin schemes*. A Souslin scheme in a metric space X is a family $(P_s \colon s \in \omega^{<\omega})$ of closed sets. A set A is analytic if and only if it can be represented as

$$A = \bigcup_{f \in \omega^\omega} \bigcap_{n \in \omega} P_{f \upharpoonright n}$$

for some Souslin scheme $(P_s \colon s \in \omega^{<\omega})$. Using a technique now known as the *increasing sets lemma*, Davies constructs a function $f : \omega \to \omega$ such that for each n, the closed set

$$F_n := \bigcup_{s \leq f \upharpoonright n} \bigcap_{i < |s|} P_{s \upharpoonright i}$$

(where $s \leq f \upharpoonright n$ means $|s| = n$ and $s(i) \leq f(i)$ for $i < n$) has sufficiently large $\mathcal{H}^s_{\varepsilon_n}$-measure, for some $\varepsilon_n > 0$. The intersection of the F_n is then the desired closed subset of positive measure. If we translate this to the canonical representation of Σ^1_1 classes in Cantor space, we obtain the following version of the Besicovitch-Davies Theorem.

Theorem 4. *For each Σ^1_1 class C of dimension $d > 0$ and for $\varepsilon > 0$, written in canonical form*

$$C = \{X \mid \exists Y \, \forall a \, \exists b \; R(X, Y, a, b)\}$$

where R is a recursive predicate, there exists a function $g \in \omega^\omega$ such that for each f majorizing g, the class

$$C_f := \{X \mid \exists Y \, \forall a \, \exists b < f(a) \; R(X, Y, a, b)\}$$

is a closed subclass of dimension at least $d - \varepsilon$.

Initially, one may think that the computational difficulty in determining whether a set of reals has *positive Hausdorff dimension* could be similar to the difficulty in determining whether it has *positive Lebesgue measure*, but we find that it is more similar to the determining whether it is *non-empty* – and this is more difficult than the measure question. While questions about Lebesgue measure can often be answered using an arithmetical oracle, for non-emptiness we often have to go beyond even the hyperarithmetical. As we shall see this level of difficulty first arises at the G_δ ($\mathbf{\Pi^0_2}$) level; we start by going over the simpler cases of open ($\mathbf{\Sigma^0_1}$), closed ($\mathbf{\Pi^0_1}$), and F_σ ($\mathbf{\Sigma^0_2}$) sets.

Theorem 5. *The following families are identical, and have Σ^0_1-complete index sets.*

1. *Σ^0_1 classes that are nonempty;*
2. *Σ^0_1 classes that have positive Hausdorff dimension;*
3. *Σ^0_1 classes that have positive measure.*

Proof. Given an c.e. set $W_e \subseteq 2^{<\omega}$, let $\mathcal{W}_e = \bigcup_{\sigma\in 2^{<\omega}} N_\sigma$, where $N_\sigma = \{X \in 2^\omega : \sigma \subset X\}$. Since any non-empty open set has positive Lebesgue measure, and having positive Lebesgue measure implies having positive dimension, the three statements are equivalent. The corresponding index sets are c.e. since $W_e \neq \emptyset$ if and only if $\exists s, \sigma(\varphi_{e,s}(\sigma)\downarrow$, and they are complete by Rice's Theorem. □

The case of Π^0_1 classes is only slightly more complicated.

Theorem 6. *The set of indices of Π^0_1 classes that are nonempty is Π^0_1-complete.*

Proof. A tree T does not have an infinite path if and only if for some level n, no string of length n is in T. If T is co-c.e. the latter event is c.e. and hence the set $\{e : [T_e] \neq \emptyset\}$ is Π^0_1. It is Π^0_1-hard by Rice's Theorem. □

Theorem 7. *The set of indices of Π^0_1 classes that have positive Lebesgue measure is Σ^0_2-complete.*

Proof (Sketch). Given a tree T_e, $[T_e]$ has positive Lebesgue measure if and only if $\exists n \forall m (|T_e \cap \{0,1\}^m| \geq 2^{m-n}|)$. Hence the corresponding index set is Σ^0_2. One can reduce the Σ^0_2 complete set $\mathrm{Fin} = \{e : W_e \text{ finite}\}$ to it by effectively building, for each e, a tree T_e such that if and only if a given W_e is finite, the measure is positive. This is achieved by cutting the measure in half (i.e. terminating an appropriate number of nodes) whenever another number enters W_e. □

Theorem 8. *The set of indices of Π^0_1 classes of Hausdorff dimension zero is Π^0_2-complete.*

Proof (Sketch). A Π^0_1 class $\mathcal{C}$ has Hausdorff dimension zero if and only if for each $d > 0$ and n, there is a clopen set U_n, induced by finitely many strings $\sigma_1, \ldots, \sigma_k$, so that $\sum 2^{-d|\sigma_i|} \leq 2^{-n}$, and such that the Σ^0_1 statement $\mathcal{C} \subseteq U_n$ holds. Thus the set of indices of Π^0_1 classes of Hausdorff dimension zero is Π^0_2. To see that this set is in fact Π^0_2-complete, we reduce the Π^0_2 complete set $\mathrm{Inf} = \{e : W_e \text{ infinite}\}$

to it. This is done by controlling the branching rate: Given e, we construct a co-r.e. tree T_e. Each time a new number enters the c.e. set, the branching rate of T_e is reduced: When we see the n-th number enter W_e at stage s, we thin out T_e by delaying, above level s, the level at which the next splitting occurs by one. $\square$

Theorem 9. *The set of indices of Σ^0_2 classes that are nonempty is Σ^0_2-complete.*

Proof. A Σ^0_2 class is nonempty if and only if some of the Π^0_1 classes in the effective union are nonempty. $\square$

Theorem 10. *The set of indices of Σ^0_2 classes of positive Hausdorff dimension is Σ^0_2-complete.*

Proof. Hausdorff dimension is *countably stable*, that is, if $E = \bigcup_n E_n$, then $\dim_{\mathrm{H}}(E) = \sup_n \dim_{\mathrm{H}}(E_n)$. Hence a Σ^0_2 class has positive Hausdorff dimension if and only if some of the Π^0_1 classes in the effective union have positive dimension. $\square$

Theorem 11. *The set of indices of Π^0_2 classes that have positive Hausdorff dimension is Σ^1_1-complete.*

Proof. For a given Π^0_2 class $\mathcal{G}$, we can consider the "Cartesian product"

$$\mathcal{P} = \mathcal{G} \times 2^\omega := \{G \oplus H \mid G \in \mathcal{G},\ H \in 2^\omega\} \tag{1}$$

where $\oplus$ denotes the usual recursion-theoretic join. By the product formula for Hausdorff dimension (adapted to Cantor space, see [17, 19]), if $\mathcal{G} \neq \emptyset$,

$$\dim_{\mathrm{H}}(\mathcal{G} \times 2^\omega) \geq \frac{\dim_{\mathrm{H}}(\mathcal{G}) + \dim_{\mathrm{H}}(2^\omega)}{2} = \frac{\dim_{\mathrm{H}}(\mathcal{G})}{2} + \frac{1}{2},$$

Hence the set $\mathcal{P}$ has positive Hausdorff dimension if and only if it has dimension at least $1/2$, if and only if $\mathcal{G} \neq \varnothing$. Since the set of indices of Π^0_2 classes in 2^ω that are nonempty is Σ^1_1-hard, so is the set of indices of Π^0_2 classes that have positive Hausdorff dimension. By the Besicovitch-Davies Theorem 4, the set of indices of Σ^1_1 classes that are of positive dimension is Σ^1_1, since

$$\dim\{X \mid \exists Y \forall a \exists b\, R(X, Y, a, b)\} > 0 \Leftrightarrow \exists f\ \dim C_f > 0.$$

$\square$

Thus, the Besicovitch-Davies Theorem (Theorem 4) turns out to be enough information to completely classify the index set complexity of classes that have positive Hausdorff dimension from arithmetical pointclasses and up to Σ^1_1; see Figure 1.

Question 2. What is the complexity of the set of indices of Π^1_1 classes that have positive Hausdorff dimension?

At the level Π^0_2 it is far more complicated to determine whether a class has positive dimension than whether it has positive Lebesgue measure (this is Σ^0_3).

Question 3. What is the complexity of the set of indices of Σ^1_1 classes that have positive Lebesgue measure?

Fig. 1: Index set complexity of some classes of reals. For example, the set of indices of Π^0_2 classes that are of positive Hausdorff dimension is Σ^1_1-complete, and this is shown in Theorem 11.

Family	Nonempty?	Positive Hausdorff dimension?
Σ^0_1	Σ^0_1-complete (5)	Σ^0_1-complete (5)
Π^0_1	Π^0_1-complete (6)	Σ^0_2-complete (8, 10)
Σ^0_2	Σ^0_2-complete (9)	
Π^0_2	Σ^1_1-complete	Σ^1_1-complete (11)
Σ^1_1		

3 Closed subsets of positive dimension

Recall that $A \leq_T B$ if A is Turing reducible to B; $A \leq_h B$ if A is hyperarithmetical in B; and, for sets of reals $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \leq_w \mathcal{B}$ if $\mathcal{A}$ is weakly (Muchnik) reducible to $\mathcal{B}$, i.e., for each $B \in \mathcal{B}$ there is some $A \in \mathcal{A}$ such that $A \leq_T B$.

Definition 1. *Let U be a Σ^1_1 class of positive Hausdorff dimension. The mass problem $S(U)$ is defined to be the collection of sets $X \subseteq \omega$ such that U has a $\Pi^0_1(X)$ subclass of positive dimension.*

We determine the difficulty of the mass problems $S(U)$ in Theorems 13, 15, and 16 below; the situation is summarized in Figure 2.

In the following definition, we are interested in the case $\Gamma = \Sigma^1_1$.

Definition 2. *A subset $\mathcal{B}$ of ω^ω is called a* basis *for a pointclass Γ if each nonempty collection of reals that belongs to Γ has a member in $\mathcal{B}$.*

Theorem 12 (Basis theorems for Σ^1_1). *Each of the following classes are bases for Σ^1_1:*

(1) $\{X \mid X \leq_T \mathcal{O}\}$*, the sets recursive in some Π^1_1 set (see Rogers* [22, XLII(b)]*);*
(2) $\{X \mid X <_h \mathcal{O}\}$*, the sets of hyperdegree strictly below $\mathcal{O}$ (Gandy* [6]*; see also Rogers* [22, XLIII(a)]*);*
(3) $\{X \mid X \not\leq_h A \ \ \& \ \ A \not\leq_h X\}$ *(where A is any given non-hyperarithmetical set) (Gandy, Kreisel, and Tait* [7]*).*

Theorem 13. *For each set $\mathcal{B}$ that is a basis for Σ^1_1 and each Σ^1_1 class U of positive dimension, there is some $X \in \mathcal{B}$ such that U has a $\Pi^0_1(X)$ subclass of positive dimension.*

Proof. Consider a Σ^1_1 class of the form

$$\{X \mid (\exists Y)(\forall a)(\exists b)R(a, b, Y, X)\}$$

and the closed subclass from Theorem 4,

$$C_f = \{X \mid (\exists Y)(\forall a)(\exists b < f(a))R(a, b, Y, X)\}$$

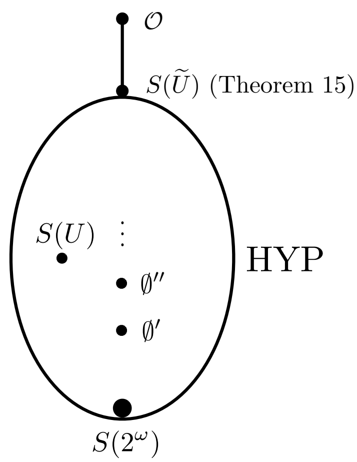

Fig. 2: The relative position in the Muchnik lattice of the various mass problems $S(U)$. At the top is Kleene's $\mathcal{O}$, according to Theorems 12(1) and 13. The ellipse represents the hyperarithmetical sets HYP with their cofinal sequence $\{0^{(\alpha)} \colon \alpha < \omega_1^{CK}\}$. The top $S(\widetilde{U})$ class is located as indicated in Theorem 16. Each $S(U)$ bounds only sets in HYP, per Theorem 15. It is not known which of the classes U here displayed might represent the set of minimal Turing degrees MIN.

which is a $\Pi_1^0(f)$ class in 2^ω. Now consider

$$\{f \in \omega^\omega \mid \dim C_f > 0\}$$

This is a Σ_2^0 class in ω^ω, in particular it is Σ_1^1, hence it has a member f in $\mathcal{B}$; and C_f for such an f is a $\Pi_1^0(G_f)$ class, where G_f is the graph of f. □

In particular, U always has a $\Pi_1^0(\mathcal{O})$ subclass of positive dimension.

Definition 3 (Solovay [26]**).** *A family F of infinite sets of natural numbers is said to be dense if each infinite set of natural numbers has a subset in F. A set A of natural numbers is said to be* recursively encodable *if the family of infinite sets in which A is recursive is dense.*

Theorem 14 (Solovay [26]**).** *The recursively encodable sets coincide with the hyperarithmetic sets.*

Theorem 15. *For each Y, if $\{Y\} \leq_w S(U)$ for some U then Y is hyperarithmetical.*

Proof. Suppose Y is recursive in each tree defining a closed subset of positive dimension of some Π_2^0 class U. By Theorem 4, Y is recursively encodable. So by Theorem 14, Y is hyperarithmetic. □

Theorem 16. *There is a Π^0_2 class $\widetilde{U}$ such that for each hyperarithmetical set Y, $\{Y\} \leq_w S(\widetilde{U})$.*

Proof. Let HYP denote the collection of all hyperarithmetical sets. Note that the class

$$U = \{X \mid \forall H \in \mathrm{HYP}\, H \leq_T X\}$$

is Σ^1_1 (an observation made by Enderton and Putnam [5]). This class U already has positive Hausdorff dimension, but the more involved proof of this fact can be avoided by again passing to the product set $\widetilde{U} = U \times 2^\omega$.

Suppose X is such that there is a $\Pi^0_1(X)$ subclass of U that is of positive dimension and hence nonempty. Then by the relativized low and hyperimmune-free basis theorems, respectively, each H in HYP is recursive in a set A that is low relative to X, and recursive in a set B that is hyperimmune-free relative to X. But A and B form a minimal pair over X, so $H \leq_T X$.

Now, U, being Σ^1_1, is the projection of a Π^0_2 class. Of course, if every member of the projection computes H then so does every member of the original Π^0_2 class (since a pair $A \oplus B$ computes both A and B). So we can replace U with such a Π^0_2 class; $\widetilde{U}$ is then still Π^0_2. □

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