

EFFECTIVE RANDOMNESS FOR CONTINUOUS MEASURES

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ABSTRACT. We investigate which infinite binary sequences (reals) are effectively random with respect to some continuous (i.e., non-atomic) probability measure. We prove that for every n , all but countably many reals are n -random for such a measure, where n indicates the arithmetical complexity of the Martin-Löf tests allowed. The proof rests upon an application of Borel determinacy. Therefore, the proof presupposes the existence of infinitely many iterates of the power set of the natural numbers. In the second part of the paper we present a metamathematical analysis showing that this assumption is indeed necessary. More precisely, there exists a computable function G such that, for any n , the statement “All but countably many reals are $G(n)$ -random with respect to a continuous probability measure” cannot be proved in ZFC_n^- . Here ZFC_n^- stands for Zermelo-Fraenkel set theory with the Axiom of Choice, where the Power Set Axiom is replaced by the existence of n -many iterates of the power set of the natural numbers. The proof of the latter fact rests on a very general obstruction to randomness, namely the presence of an internal definability structure.

1. INTRODUCTION

The goal of this paper is study under what circumstances an infinite binary sequence (real) is random with respect to some probability measure. We use the framework of Martin-Löf randomness to investigate this question. Given a measure μ , a Martin-Löf test is an effectively presented G_δ μ -nullset in which the measure of the open sets converges effectively to zero. As there are only countably many such tests, only measure-zero many reals can be covered by a Martin-Löf test for μ . The reals that cannot be covered are called Martin-Löf random for μ . Obviously, if a real X is an atom of a measure μ , then X is random for μ . If we rule out this trivial way of being random, the task becomes harder: Given a real X , does there exist a probability measure

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on the space of all infinite binary sequences such that X is not an atom of μ but X is μ -random?

In [41], we were able to show that if a real X is not computable, then such a measure exists. It is not hard to see that if a real X is computable, then the only way that X is random with respect to a measure μ is for it to be an atom of μ . Hence having non-trivial random content (with respect to any measure at all) in the sense of Martin-Löf is equivalent to being non-computable. Besides Martin-Löf randomness, various other notions of algorithmic randomness have been thoroughly investigated, such as Schnorr randomness or Kurtz randomness. Two recent books on algorithmic randomness [9, 36] provide a good overview over the various concepts. They all have in common that they use algorithmic features to separate non-randomness from randomness. Moreover, in terms of the arithmetic hierarchy, the complexities of the underlying test notions usually fall within two or three quantifiers of each other.

This suggests that in order to study the random content of a real from the point of view of algorithmic randomness in general, we should look at how this content behaves when making tests more powerful by giving them access to oracles (or equivalently, considering nullsets whose definitions are more complicated). For Martin-Löf tests, this means the test has to be effectively G_δ only in some parameter Z . This enlarges the family of admissible nullsets and, correspondingly, shrinks the set of random reals. If the parameter Z is an instance $\emptyset^{(n)}$ of the Turing jump, i.e., Z is real that can decide all Σ_n statements about arithmetic, we speak of $(n + 1)$ -randomness.

Our goal is to understand the nature of the set of reals that are not n -random with respect to *any continuous probability measure*. In particular, we want to understand how this set behaves as n grows larger (and more reals will have this property).

The restriction to continuous measures makes sense for the following reasons. By a result of Haken [17, Theorem 5], if a real is n -random, $n > 2$, with respect to some (not necessarily continuous) probability measure and not an atom of the measure, it is $(n - 2)$ -random with respect to a continuous probability measure. Thus considering arbitrary probability measures would only shift the question of how random a real is by a couple of quantifiers. And the core problem of finding a measure that makes a real random without making the real an atom of the measure remains. While we ignore features of randomness for arbitrary measures at lower levels, we develop insights into

randomness for continuous measures. At the level of 1-randomness, there is an interesting connection with computability theory: In [41], drawing on a result of Woodin [47], we showed that if a real X is not hyperarithmetical, then there exists a continuous probability measure for which X is 1-random.

Our first main result concerns the size of the set of reals that are not n -random with respect to any continuous measure. The case $n = 1$ follows of course from the result in [41] mentioned above.

Theorem 1. *For any $n \in \omega$, all but countably many reals are n -random with respect to some continuous probability measure.*

The proof features a metamathematical argument. Let us denote by NCR_n the set of all reals that are not n -random with respect to any continuous probability measure. We show that for each n , NCR_n is contained in a countable model of a fragment of set theory. More precisely, this fragment is ZFC_{n+3}^- , where ZFC_{n+3}^- denotes the axioms of Zermelo-Fraenkel set theory with the Axiom of Choice, with the power set axiom replaced by a sentence that assures the existence of $n + 3$ iterates of the power set of the natural numbers.

One may wonder whether this metamathematical argument is really necessary to prove the countability of a set of reals, in particular, whether one needs the existence of infinitely many iterates of the power set of ω to prove Theorem 1, a result about sets of reals. It turns out that this is indeed the case. This is the subject of our second main result.

Theorem 2. *There exists a computable function $G(n)$ such that for every $n \in \omega$, the statement*

“There exist only countably many reals that are not $G(n)$ -random with respect to some continuous probability measure.”

is not provable in ZFC_n^- .

This metamathematical property of NCR is reminiscent of Borel determinacy [32]. Even before Martin proved that every Borel game is determined, Friedman [11] had shown that any proof of Borel determinacy had to use uncountably many iterates of the power set of ω . Borel determinacy is a main ingredient in our proof of Theorem 1. Theorem 2 establishes that this use is, in a certain sense, inevitable.

Theorem 2 is proved via a fine structure analysis of the countable models used to show NCR_n is countable. These models are certain levels L_β of

Gödel’s constructible hierarchy. In these L_β (or rather Jensen’s version, the J -hierarchy) we exhibit sequences of non-random reals with Turing degrees cofinal among those of the model. These reals are *master codes* [3, 22], reals that code initial segments of the J -hierarchy in a way that arithmetically reflects the strong stratification of L . The main feature of this proof is a very general principle that manifests itself in various forms: an internal stratified definability structure forms a strong obstruction to randomness. This principle works for both iterated Turing jumps as well as certain levels of the J -hierarchy.

Before we proceed, we make one more comment on the restriction to continuous measures. Note that Theorem 1 is a stronger statement for continuous measures than for arbitrary measures. Furthermore, by Haken’s result [17], Theorem 2 holds for arbitrary measures if we replace $G(n)$ by $G(n) + 2$.

The paper is organized as follows. In Section 2, we introduce effective randomness for arbitrary (continuous) probability measures. We also prove some fundamental facts on randomness. In particular, we will give various ways to obtain reals that are random for *some* continuous measure from standard Martin-Löf random reals (i.e., random with respect to Lebesgue measure). We also consider the definability strength of random reals. Section 3 features the proof that for any n , all but countably many reals are n -random with respect to some continuous measure (Theorem 1). Finally, Section 4 is devoted to the metamathematical analysis of Theorem 1. In particular, it contains a proof of Theorem 2.

We expect the reader to have basic knowledge in mathematical logic and computability theory, including some familiarity with forcing, the constructible universe, and the recursion theoretic hierarchies.

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2. RANDOMNESS FOR CONTINUOUS MEASURES

In this section we review effective randomness on Cantor space 2^ω for arbitrary probability measures. We then prove some preliminary facts about random reals.

The *Cantor space* 2^ω is the set of all infinite binary sequences, also called *reals*. The topology generated by the *cylinder sets*

$$[[\sigma]] = \{x : x \upharpoonright_{|\sigma|} = \sigma\},$$

where σ is a finite binary sequence, turns 2^ω into a compact Polish space. $2^{<\omega}$ denotes the set of all finite binary sequences. If $\sigma, \tau \in 2^{<\omega}$, we use \subseteq to denote the usual prefix partial ordering. This extends in a natural way to $2^{<\omega} \cup 2^\omega$. Thus, $x \in [[\sigma]]$ if and only if $\sigma \subset x$. Finally, given $U \subseteq 2^{<\omega}$, we write $[[U]]$ to denote the open set induced by U , i.e. $[[U]] = \bigcup_{\sigma \in U} [[\sigma]]$.

2.1. Turing functionals. While the concept of a Turing functional is standard, we will later define a forcing partial order based on it, and for this purpose we give a rather complete formal definition here. The definition follows [46], with the one difference that we require Turing functionals to be recursively enumerable.

A *Turing functional* Φ is a computably enumerable set of triples (m, k, σ) such that m is a natural number, k is either 0 or 1, and σ is a finite binary sequence. Further, for all m , for all k_1 and k_2 , and for all compatible σ_1 and σ_2 , if $(m, k_1, \sigma_1) \in \Phi$ and $(m, k_2, \sigma_2) \in \Phi$, then $k_1 = k_2$ and $\sigma_1 = \sigma_2$.

We will refer to a triple (m, k, σ) as a *computation* in Φ , and we will say it is a *computation along X* when every X is an extension of σ .

In the following, we will also assume that Turing functionals Φ are *use-monotone*, which means the following hold.

- (1) For all (m_1, k_1, σ_1) and (m_2, k_2, σ_2) in Φ , if σ_1 is a proper initial segment of σ_2 , then m_1 is less than m_2 .
- (2) For all m_1 and m_2 , k_2 and σ_2 , if $m_2 > m_1$ and $(m_2, k_2, \sigma_2) \in \Phi$, then there are k_1 and σ_1 such that $\sigma_1 \subseteq \sigma_2$ and $(m_1, k_1, \sigma_1) \in \Phi$.

We write $\Phi^\sigma(m) = k$ to indicate that there is a τ such that τ is an initial segment of σ , possibly equal to σ , and $(m, k, \tau) \in \Phi$. In this case, we also write $\Phi^\sigma(m) \downarrow$, as opposed to $\Phi^\sigma(m) \uparrow$, indicating that for all k and all $\tau \subseteq \sigma$, $(m, k, \tau) \notin \Phi$. If, moreover, (m, k, τ) is enumerated into Φ by time s , we write $\Phi_s^\sigma(m) = k$.

If $X \in 2^\omega$, we write $\Phi^X(m) = k$ (and $\Phi_s^X(m) = k$, respectively) to indicate that there is an l such that $\Phi^{X \upharpoonright l}(m) = k$ (and this is enumerated by time s , respectively). This way, for given $X \in 2^\omega$, Φ^X defines a partial function from

ω to $\{0, 1\}$ (identifying reals with sets of natural numbers). If this function is total, it defines a real Y , and in this case we write $\Phi(X) = Y$ and say that Y is Turing reducible to X via Φ , $Y \leq_T X$.

By use-monotonicity, if $\Phi^\sigma(m) \downarrow$, then $\Phi^\sigma(n) \downarrow$ for all $n < m$. If we let \bar{m} be maximal such that $\Phi^\sigma(\bar{m}) \downarrow$, Φ^σ gives rise to a string τ of length $\bar{m} + 1$,

$$\tau = \Phi^\sigma(0) \dots \Phi^\sigma(\bar{m}).$$

If $\Phi^\sigma(n) \uparrow$ for all n , we put $\tau = \emptyset$. On the other hand, if \bar{m} does not exist, then Φ^σ gives rise to a real Y . We write $\Phi(\sigma) = \tau$ or $\Phi(\sigma) = Y$, respectively. This way a Turing functional induces a function from $2^{<\omega}$ to $2^{<\omega} \cup 2^\omega$ that is *monotone*, that is, $\sigma \subseteq \tau$ implies $\Phi(\sigma) \subseteq \Phi(\tau)$. Note that $\Phi(\sigma)$ is not necessarily a computable function, but we can effectively approximate it by prefixes. More precisely, there exists a computable mapping $(\sigma, s) \mapsto \Phi_s(\sigma)$ so that $\Phi_s(\sigma) \subseteq \Phi_{s+1}(\sigma)$, $\Phi_s(\sigma) \subseteq \Phi_s(\sigma \frown i)$ for $i \in \{0, 1\}$, and $\lim_s \Phi_s(\sigma) = \Phi(\sigma)$.

If, for a real X , $\lim_n |\Phi(X \upharpoonright_n)| = \infty$, then $\Phi(X) = Y$, where Y is the unique real that extends all $\Phi(X \upharpoonright_n)$. In this way, Φ also induces a partial, continuous function from 2^ω to 2^ω . We will use the same symbol Φ for the Turing functional, the monotone function from $2^{<\omega}$ to $2^{<\omega}$, and the partial, continuous function from 2^ω to 2^ω . It will be clear from the context which Φ is meant. Φ is called *total* if $\Phi(X)$ is a real for all $X \in 2^\omega$. If Φ is total and $\Phi(X) = Y$, then Y is called *truth-table reducible* to X , $Y \leq_{tt} X$.

Turing functionals can be relativized with respect to a parameter Z , by requiring that Φ is r.e. in Z . We call such functionals *Turing Z -functionals*. This way we can consider relativized Turing reductions. A real X is Turing reducible to a real Y relative to a real Z , written $X \leq_{T(Z)} Y$, if there exists a Turing Z -functional Φ such that $\Phi(X) = Y$.

2.2. Probability measures. By the Carathéodory extension theorem, a Borel probability measure μ on 2^ω is completely specified by its values on *clopen sets*, i.e., on finite unions of basic open cylinders. In particular, $\mu[\emptyset] = 1$, and the additivity of μ implies that for all $\sigma \in 2^{<\omega}$,

$$(2.1) \quad \mu[\sigma] = \mu[\sigma \frown 0] + \mu[\sigma \frown 1].$$

An *additive premeasure* is a function $\eta : 2^{<\omega} \rightarrow \mathbb{R}^{\geq 0}$ with $\eta(\emptyset) = 1$ and $\eta(\sigma) = \eta(\sigma \frown 0) + \eta(\sigma \frown 1)$ for all $\sigma \in 2^{<\omega}$. Any additive premeasure induces a Borel probability measure, and if we restrict a Borel probability measure μ to its values on cylinders, we obtain an additive premeasure whose Carathéodory

extension is μ . We can therefore identify a Borel probability measure on 2^ω with the additive premeasure it induces. We will exclusively deal with Borel probability measures and in the following simply write *measure* to denote a Borel probability measure on 2^ω .

The *Lebesgue measure* λ on 2^ω is obtained by distributing a unit mass uniformly along the paths of 2^ω , i.e., by setting $\lambda[\sigma] = 2^{-|\sigma|}$. A *Dirac measure*, on the other hand, is defined by putting a unit mass on a single real, i.e., for $X \in 2^\omega$, let

$$\delta_X[\sigma] = \begin{cases} 1 & \text{if } \sigma \subset X, \\ 0 & \text{otherwise.} \end{cases}$$

If, for a measure μ and $X \in 2^\omega$, $\mu(\{X\}) > 0$, then X is called an *atom* of μ . Obviously, X is an atom of δ_X . A measure that does not have any atoms is called *continuous*.

2.3. Representation of measures and Martin-Löf randomness. To incorporate measures into an effective test for randomness we represent them as reals. This can be done in various ways (for example, identify them with the underlying premeasure and code that), but in order for the main arguments in [41] to work, the representation has to reflect some of the topological properties of the space of probability measures.

Let $\mathcal{M}(2^\omega)$ be the set of all Borel probability measures on 2^ω . With the weak-* topology, this becomes a compact Polish space (see [26, Theorem 17.23]). It is possible to choose a countable dense subset $\mathcal{D} \subseteq \mathcal{M}(2^\omega)$ so that every measure in $\mathcal{M}(2^\omega)$ is the limit of an effectively converging Cauchy sequence of measures in \mathcal{D} . Moreover, the structure of the measures in \mathcal{D} is such that they give rise to a canonical continuous surjection $\rho : 2^\omega \rightarrow \mathcal{M}(2^\omega)$ with the additional property that for every $R \in 2^\omega$, $\rho^{-1}(\{\rho(R)\})$ is $\Pi_1^0(R)$. For details on the construction of ρ , see [4, Section 2]. If $\rho(R) = \mu$, then R is called a *representation* of μ . A measure may have several distinct representations with respect to ρ . If μ is given, R_μ will always denote a representation of μ .

Working with representations, we can apply computability theoretic notions to measures. The following two observations appeared as Propositions 2.2 and 2.3, respectively, in [41].

Proposition 2.1. *Let $R \in 2^\omega$ be a representation of a measure $\mu \in \mathcal{M}(2^\omega)$. Then the relations*

$$\mu[\sigma] < q \quad \text{and} \quad \mu[\sigma] > q \quad (\sigma \in 2^{<\omega}, q \in \mathbb{Q})$$

are r.e. in R .

It follows that the representation of a measure can effectively approximate its values on cylinders to arbitrary precision.

Proposition 2.2. *Let $R \in 2^\omega$ be a representation of a measure $\mu \in \mathcal{M}(2^\omega)$. Then R computes a function $g_\mu : 2^{<\omega} \times \omega \rightarrow \mathbb{Q}$ such that for all $\sigma \in 2^{<\omega}$, $n \in \omega$,*

$$|g_\mu(\sigma, n) - \mu[\sigma]| \leq 2^{-n}.$$

We say a real X is *recursive in μ* if $X \leq_T R_\mu$ for every representation R_μ of μ . On the other hand, we say a real computes a measure if it computes *some* representation of it. A measure does not necessarily have a representation of least Turing degree [4, Theorem 4.2].

We will later show that the question of whether a real is random with respect to a continuous measure can be reduced to considering only continuous dyadic measures. A measure μ is *dyadic* if every measure of a cylinder is of the form $\mu[\sigma] = m/2^n$ with m, n non-negative integers.

For dyadic measures, it makes sense to speak of *exact computability*: A dyadic measure μ is *exactly computable* if the function $\sigma \mapsto \mu[\sigma]$ is a computable mapping from $2^{<\omega}$ to \mathbb{Q} . Note that for exactly computable measures, the relation $\mu[\sigma] > q$, q rational, is decidable, whereas in the general case for μ with a computable representation we only know it is Σ_1^0 .

If we encode a dyadic measure μ by collecting the ternary expansions of its values on cylinders in a single real Y , we obtain a representation not in the sense of ρ , but that is minimal in the following sense: Any real that can compute an approximation function to μ in the sense of Proposition 2.2 can compute Y .

We can now give the definition of a general Martin-Löf test. The definition is a generalization of Martin-Löf n -tests and Martin-Löf n -randomness for Lebesgue measure. We relativize both with respect to a representation of the measure and an additional parameter.

Definition 2.3. Suppose μ is a probability measure on 2^ω , and R is a representation of μ . Suppose further that $Z \in 2^\omega$ and $n \geq 1$.

- (1) An (R, Z, n) -test is a set $W \subseteq \omega \times 2^{<\omega}$ which is recursively enumerable in $(R \oplus Z)^{(n-1)}$, the $(n-1)$ st Turing jump of $R \oplus Z$ such that

$$\sum_{\sigma \in W_n} \mu[\sigma] \leq 2^{-n},$$

where $W_n = \{\sigma : (n, \sigma) \in W\}$

- (2) A real X passes a test W if $X \notin \bigcap_n [W_n]$. If X does not pass a test W , we also say X is covered by W (or (W_n) , respectively).
- (3) A real X is (R, Z, n) -random if it passes all (R, Z, n) -tests.
- (4) A real X is Martin-Löf n -random for μ relative to Z , or simply (μ, Z, n) -random if there exists a representation R_μ such that X is (R_μ, Z, n) -random. In this case we say R_μ witnesses the μ -randomness of X .

If the underlying measure is Lebesgue measure λ , we often drop reference to the measure and simply say X is (Z, n) -random. We also drop the index 1 in case of $(\mu, Z, 1)$ -randomness and simply speak of μ -randomness relative to Z or μ - Z -randomness. If $Z = \emptyset$, on the other hand, we speak of (μ, n) - or μ - n -randomness. Note also that if μ is Z -computable, say $R_\mu \leq_T Z$, then (R_μ, Z, n) -randomness is the same as $(R_\mu, Z^{(n-1)}, 1)$ -randomness.

Remark 2.4. The original definition of n -randomness for Lebesgue measure λ given by Kurtz [29] uses tests based on Σ_n classes. However, it is possible to approximate Σ_n^0 classes from outside in measure by open sets. Kurtz [29] and Kautz [25] showed that such an approximation in measure can be done effectively for classes of the lightface finite Borel hierarchy, in the sense that a Σ_n^0 class can be approximated in measure by a $\Sigma_1^{0, \emptyset^{(n-1)}}$ class. Therefore, while the definitions based on Σ_n^0 nullsets and on $\Sigma_1^{0, \emptyset^{(n-1)}}$ nullsets do not give the same notion of test, they yield the same class of random reals (see [9, Section 6.8] for a complete presentation of this argument).

The proof that the two approaches yield the same notion of n -randomness relativizes. Moreover, the approximation in measure by open sets is possible for any Borel probability measure, as any finite Borel measure on a metric space is regular. Finally, using Proposition 2.1 one can show inductively that, given a representation R of μ , the relations $\mu(S) > q$ and $\mu(S) < q$ (for q rational) are uniformly $\Sigma_n^{0, R}$ for any Σ_n^0 class S . The latter fact is a key ingredient in the equivalence proof. Therefore, (μ, Z, n) -randomness could alternatively be defined via $\Sigma_n^{0, R \oplus Z}$ tests. We prefer the approach given in

Definition 2.3, because open sets are usually easier to work with, and because most techniques relativize.

Levin [30] introduced the alternative concept of a *uniform test* for randomness, which is representation-independent (see also [13]). Day and Miller [4, Theorem 1.6] have shown that for any measure μ and for any real X , X is μ -random in the sense of Definition 2.3 if and only if X is μ -random for uniform tests.

Since, for fixed R_μ , Z , and n , there are only countably many (R_μ, Z, n) -tests, it follows from countable additivity that the set of (μ, Z, n) -random reals for any μ and any Z has μ -measure 1. Hence there always exist (μ, Z, n) -random reals for any measure μ , any real Z , and any $n \geq 1$.

However, μ , Z , and n put some immediate restrictions on the relative definability of any (μ, Z, n) -random real.

Proposition 2.5. *If μ is a continuous measure, X is (μ, Z, n) -random via a representation R_μ , then X cannot be $\Delta_n^0(R_\mu \oplus Z)$.*

Proof. If X is $\Delta_n^0(R_\mu \oplus Z)$, then $X \leq_{\text{T}} (R_\mu \oplus Z)^{(n-1)}$, and we can build a (μ, Z, n) -test covering X by using the cylinders given by its initial segments. \square

It is also immediate from the definition of randomness that any atom of a measure is random with respect to it. This is a trivial way for a real to be random. The proposition below (a straightforward relativization of a result by Levin [48], see also [41, Proposition 3.3]) shows that atoms of a measure are also computationally trivial (relative to the measure).

Proposition 2.6 (Levin). *If for a measure μ and a real X , $\mu\{X\} > 0$, then $X \leq_{\text{T}} R_\mu$ for any representation R_μ of μ .*

Since we are interested in randomness for continuous measures, the case of atomic randomness is excluded a priori.

2.4. Image measures and transformation of randomness. Let $f : 2^\omega \rightarrow 2^\omega$ be a Borel measurable function. If μ is a measure on 2^ω , the *image measure* μ_f is defined by

$$\mu_f(A) = \mu(f^{-1}(A)).$$

It can be shown that every probability measure is the image measure of Lebesgue measure λ for some f . Oxtoby [37, Theorem 1] proved that any

continuous, positive probability measure on 2^ω can be transformed into Lebesgue measure on 2^ω via a homeomorphism. Here, a measure is *positive* if $\mu[\sigma] > 0$ for $\sigma \in 2^{<\omega}$.

Levin (see [48, Theorem 4.3]), and independently Kautz [25, Corollary IV.3.18] (see also [2]) proved effective analogs of these results. For a computable measure μ on 2^ω there exists a Turing functional Φ defined on almost every real such that μ is the image measure of λ under Φ . If μ is, moreover, continuous and positive, then Φ has an inverse that transforms μ into λ .

A consequence of the Levin-Kautz theorem is that every non-recursive real that is random with respect to a computable probability measure is Turing equivalent to a λ -random real. We will show now that for continuous measures, this can be strengthened to *truth-table equivalence*.

Proposition 2.7. *Let X be a real. For any $Z \in 2^\omega$ and any $n \geq 1$, the following are equivalent.*

- (i) X is (μ, Z, n) -random for a continuous measure μ recursive in Z .
- (ii) X is (ν, Z, n) -random for a continuous, positive, dyadic measure ν exactly computable in Z .
- (iii) There exists a Turing Z -functional Φ such that Φ is an order-preserving homeomorphism of 2^ω , and $\Phi(X)$ is (λ, Z, n) -random.
- (iv) X is truth-table equivalent relative to Z to a (λ, Z, n) -random real.

Here, the order on 2^ω is the *lexicographical order* given by

$$X < Y \quad :\Leftrightarrow \quad X(N) < Y(N) \quad \text{where } N = \min\{n: X(n) \neq Y(n)\}.$$

Proof. We give a proof for $Z = \emptyset$ and $n = 1$. It is routine to check that the proof relativizes and generalizes to higher levels of randomness.

(i) \Rightarrow (ii): Let X be μ -random, where μ is a continuous, computable measure. We construct a continuous, positive, dyadic, and exactly computable measure ν such that X is random with respect to ν , too. The construction is similar to Schnorr's rationalization of martingales [44] (see also [9, Proposition 7.1.2]).

We define ν by recursion on the full binary tree $2^{<\omega}$. To initialize, let $\nu^*[\emptyset] = 2$. Now assume $\nu^*[\sigma]$ is defined such that,

$$\mu[\sigma] < \nu^*[\sigma] < \mu[\sigma] + 2^{-|\sigma|+1}.$$

A simple case distinction shows that

$$\max\{\mu[\sigma \frown 0], \nu^*[\sigma] - \mu[\sigma \frown 1] - 2^{-|\sigma|}\} < \min\{\mu[\sigma \frown 0] + 2^{-|\sigma|}, \nu^*[\sigma] - \mu[\sigma \frown 1]\}.$$

As the dyadic rationals are dense in \mathbb{R} , there exists a dyadic rational r in this interval, and by Proposition 2.2 we can find such an r effectively in σ . Put

$$\nu^*[\sigma \frown 0] = r, \quad \nu^*[\sigma \frown 1] = \nu^*[\sigma] - r.$$

Then clearly

$$\nu^*[\sigma \frown 0] + \nu^*[\sigma \frown 1] = \nu^*[\sigma],$$

and by the choice of r ,

$$\begin{aligned} \mu[\sigma \frown 0] &< \nu^*[\sigma \frown 0] < \mu[\sigma \frown 0] + 2^{-(|\sigma|)}, \\ \mu[\sigma \frown 1] &< \nu^*[\sigma \frown 1] < \mu[\sigma \frown 1] + 2^{-(|\sigma|)}. \end{aligned}$$

We normalize by letting $\nu = \nu^*/2$. By construction of ν^* , the measure ν is dyadic and exactly computable. It is also clear from the construction that for all σ , $\mu[\sigma] < 2\nu[\sigma]$. In particular, ν is positive. Finally, if (V_n) is a test for ν , by letting $W_n = V_{n+1}$ we obtain a μ -test that covers every real covered by (V_n) . Hence if X is μ -random, then X is also ν -random.

(ii) \Rightarrow (iii): Suppose ν is an exactly computable, continuous, positive, dyadic measure. Since ν is continuous and 2^ω is compact, for every m there exists a least $l_m \in \omega$ such that whenever $|\sigma| \geq l_m$, then $\nu[\sigma] \leq 2^{-m}$. Without loss of generality, we can assume that $l_m < l_{m+1}$. As ν is exactly computable, the mapping $m \mapsto l_m$ is computable.

We define inductively a mapping $\varphi : 2^{<\omega} \rightarrow 2^{<\omega}$ that will induce the desired homeomorphism. In order to do so, we first define, for every $\tau \in 2^{<\omega}$, an auxiliary finite, non-empty set $E_\tau \subseteq 2^{<\omega}$. It will hold that

- (a) all strings in E_τ are of length l_{2n} , where n is the length of τ ;
- (b) if $\sigma \subseteq \tau$, then every string in E_τ is an extension of some string in E_σ ;
- (c) if σ and τ are incomparable, then E_σ and E_τ are disjoint; moreover, if $|\sigma| = |\tau|$ and σ is lexicographically less than τ , then all strings in E_σ are lexicographically less than any string in E_τ ;
- (d) for all n , $\bigcup_{|\tau|=n} E_\tau = 2^\omega$;
- (e) for all τ , $0 < \nu[E_\tau] \leq 2^{-|\tau|}(2 - 2^{-|\tau|})$.

Put $E_\emptyset = \{\emptyset\}$. Suppose now that E_τ is defined for all strings τ of length at most n , and that for these sets E_τ , (a)-(e) are satisfied.

Given any τ of length n , let

$$F_\tau = \{\sigma : |\sigma| = l_{2(n+1)} \text{ \& } \sigma \text{ extends some string in } E_\tau\}.$$

Find the least (with respect to the usual lexicographic ordering) $\sigma \in F_\tau$ such that

$$\sum_{\substack{\eta \leq \sigma \\ \eta \in F_\tau}} \nu[\eta] \geq \nu[E_\tau]/2.$$

Let $E_{\tau \frown 0} = \{\eta \in F_\tau : \eta < \sigma\}$ and put the remaining strings of F_τ into $E_{\tau \frown 1}$. This ensures that $E_{\tau \frown 0}$ and $E_{\tau \frown 1}$ satisfy (a), (b), (c), and (d). Moreover, by the choice of the length of strings in F_τ and property (e) for E_τ , both $E_{\tau \frown 0}$ and $E_{\tau \frown 1}$ are non-empty. As ν is positive, this implies $\nu[[E_{\tau \frown i}]] > 0$ for each $i \in \{0, 1\}$. Moreover, for each $i \in \{0, 1\}$, we can use the induction hypothesis for E_τ and deduce that

$$\begin{aligned} \nu[[E_{\tau \frown i}]] &\leq \nu[[E_\tau]]/2 + 2^{-2|\tau|-1} \\ &\leq 2^{-|\tau|-1}(2 - 2^{-|\tau|}) + 2^{2(|\tau|+1)} \\ &= 2^{-|\tau|-1}(2 - 2^{-|\tau|-1}), \end{aligned}$$

which yields the bound in (d).

Now we define the mapping φ : Put $\varphi(\emptyset) = \emptyset$. Suppose now $\varphi(\sigma)$ is defined for all τ of length less than or equal to n . Given τ of length n , map all strings in $E_{\tau \frown 0}$ to $\tau \frown 0$, and all strings in $E_{\tau \frown 1}$ to $\tau \frown 1$. To make φ defined on all strings, map any string that extends some string in E_τ but is a true prefix of some string in F_τ to τ .

It is clear from the construction that φ induces a total, order preserving mapping $\Phi : 2^\omega \rightarrow 2^\omega$ by letting

$$\Phi(X) = \lim_n \varphi(X \upharpoonright_n).$$

Φ is onto since for every σ , E_σ is not empty. We claim that Φ is also one-to-one. Suppose $\Phi(X) = \Phi(Y)$. This implies that for all n , $\varphi(X \upharpoonright_n) = \varphi(Y \upharpoonright_n)$, that is, for all n , $X \upharpoonright_n$ and $Y \upharpoonright_n$ belong to the same E_σ . Since ν is positive, the diameter of the E_σ goes to 0 along any path. Hence $X = Y$.

It remains to show that $\Phi(X)$ is Martin-Löf random. Suppose not, then there exists a λ -test (W_n) that covers $\Phi(X)$. Let

$$V_n = \bigcup_{\sigma \in W_{n+2}} E_\sigma.$$

Then (V_n) covers X . Furthermore, the (V_n) are uniformly enumerable since the mapping $\sigma \mapsto E_\sigma$ is computable by the construction of the E_σ . Finally,

$$\sum_{\tau \in V_n} \nu[[\tau]] = \sum_{\sigma \in W_{n+2}} \nu[[E_\sigma]] \leq \sum_{\sigma \in W_{n+2}} 2^{-|\sigma|+2} \leq 2^{-n}.$$

thus X is not ν -random, contradiction.

(iii) \Rightarrow (iv): This is immediate.

(iv) \Rightarrow (i): This follows from Theorem 5.7 in [41].

□

The result also suggests that if we are only interested in whether a real is random with respect to a continuous measure, representational issues do not really arise. We can restrict ourselves to dyadic measures, which have a minimal representation.

Remark 2.8. We will henceforth, unless explicitly noted, assume that *any measure is a dyadic measure*. With respect to effectiveness considerations, we identify R_μ with μ and write μ' instead of R'_μ and so forth.

2.5. Continuous randomness via Turing reductions. While Proposition 2.7 gives a necessary and sufficient criterion for reals being random for a continuous measure, we will later need further techniques to show that a given real is random with respect to a continuous measure. As many of our arguments will involve arithmetic definability, it will be helpful to know to what extent randomness for continuous measures can be “transferred” via Turing reductions instead of truth-table reductions. The key ingredients are a theorem by Demuth [6] and a result by Kurtz [29].

Demuth [6, Theorem 17] showed that every non-recursive real truth-table below a Martin-Löf random real measure is Turing equivalent to a Martin-Löf-random real. The proof relativizes (as can be seen from the presentation in [9, Theorems 6.12.9 and 8.6.1]) and yields the next proposition.

Recall that we only consider dyadic measures and hence drop reference to a representation. Nevertheless, the results in this section are not dependent on the existence of a minimal representation and can be reformulated accordingly.

Proposition 2.9 (Demuth). *Suppose Y is (μ, Z, n) -random ($n \geq 1$) and X is truth-table reducible to Y relative to $(\mu \oplus Z)^{(k)}$ for some $k \leq n - 1$ (i.e., $X \leq_{\text{tt}((\mu \oplus Z)^{(k)})} Y$). Further suppose X is not recursive in $(\mu \oplus Z)^{(k)}$. Then X is Turing equivalent relative to $(\mu \oplus Z)^{(k)}$ to a $(\lambda, \mu \oplus Z, n)$ -random real.*

Kurtz [29, Theorem 4.3] observed that 2-random reals are \emptyset' -dominated. More precisely, there exists a \emptyset' -computable function dominating every function computable from a 2-random real.

The proof is based on the following idea (see [36, Proposition 5.6.28]): Given a Turing functional Φ , \emptyset' can decide, given rational q and $n \in \omega$,

whether

$$\lambda \{Y : \Phi(Y)(k) \text{ is defined for all } k \leq n\} > q.$$

For each n , let q_n be maximal of the form $i \cdot 2^{-n}$ so that the above holds, and let t_n be such that Φ converges on at least measure q_n -many strings of length t_n by time t_n . Construct a function $f \leq_{\mathbb{T}} \emptyset'$ such that $f(n)$ dominates all function values $\Phi(Y)$ computed with use t_n and within t_n steps. Then the set of all Y for which $\Phi(Y)$ is total and not dominated by f has Lebesgue measure 0 and can be captured by a \emptyset' -Martin-Löf-test. The argument relativizes to other measures and parameters, and we obtain the following.

Proposition 2.10 (Kurtz). *Given a measure μ and a real Z , there exists a function $f \leq_{\mathbb{T}} (\mu \oplus Z)'$ such that for every $(\mu, Z, 2)$ -random X , if $g \leq_{\mathbb{T}(Z \oplus \mu)} X$, then g is dominated by f .*

Together with Proposition 2.9 this yields a sufficient criterion for continuous randomness.

Lemma 2.11. *Suppose $n \geq 3$ and Y is (μ, Z, n) -random. If $X \leq_{\mathbb{T}(\mu \oplus Z)} Y$ and $X \not\leq_{\mathbb{T}} (\mu \oplus Z)'$, then X is $(\nu, (\mu \oplus Z)'', n-2)$ -random for some continuous measure $\nu \leq_{\mathbb{T}} (\mu \oplus Z)''$.*

Proof. We assume $Z = \emptyset$ to keep notation simple. Suppose $X \leq_{\mathbb{T}(\mu)} Y$ via a Turing reduction Φ . By Proposition 2.10, the use and the convergence time of Φ on Y are dominated by some function recursive in μ' . We can modify Φ to $\tilde{\Phi}$ such that $\tilde{\Phi}$ is a truth-table reduction relative to μ' and $\tilde{\Phi}(Y) = X$.

By Proposition 2.9, X is Turing equivalent relative to μ' to a (λ, μ, n) -random real R . Any (λ, μ, n) -random real is also $(\lambda, \mu', n-1)$ -random, and so we can apply Proposition 2.10 to X and R to conclude that they are truth-table equivalent relative to μ'' . This in turn means that X is truth-table equivalent relative to μ'' to a $(\lambda, \mu'', n-2)$ -random real, which by Proposition 2.7 implies that X is $(\nu, \mu'', n-2)$ -random for a continuous measure recursive in μ'' . \square

2.6. The definability strength of randomness. Lemma 2.11 shows that sufficiently high randomness for continuous measures propagates downward under Turing reductions (losing some of the randomness strength, however). This result was partly based on the fact that, for $n \geq 2$, n -random reals cannot compute fast-growing functions (beyond what is computable by \emptyset'). There is further evidence that the computational strength of n -random reals is rather limited.

For example, random reals are generalized low (relative to the measure). This is a generalization of a result due to Kautz [25, Theorem III.2.1].

Proposition 2.12 (Kautz). *Let μ be a continuous measure, and suppose X is μ -($n + 1$)-random, where $n \geq 1$. Then*

$$(X \oplus \mu)^{(n)} \equiv_{\text{T}} X \oplus \mu^{(n)}$$

The generalization works for the same reasons that n -randomness can be defined equivalently in terms of $\Sigma_n^{0,\mu}$ -tests or $\Sigma_1^{0,\mu^{(n-1)}}$ -tests: Borel probability measures are regular, and if S is a Σ_n^0 class, the relations $\mu(S) > q$ and $\mu(S) < q$ (for q rational) are uniformly $\Sigma_n^{0,\mu}$ and $\Pi_n^{0,\mu}$, respectively.

Furthermore, one can generalize a result of Downey, Nies, Weber, and Yu [8], who show that every weakly 2-random real forms a minimal pair with $0'$. This will be of central importance in Section 4. For our purposes, it suffices to consider randomness instead of weak randomness, which we do in the following lemma.

Lemma 2.13. *Suppose μ is a continuous measure and Y is μ - n -random, $n \geq 2$. If $X \leq_{\text{T}} \mu^{(n-1)}$ and $X \leq_{\text{T}} Y \oplus \mu$, then $X \leq_{\text{T}} \mu$.*

The structure of the proof is as follows: Following Downey, Nies, Weber, and Yu, we first show that the upper cone by Φ is Π_2^0 (relative to $\mu^{(n-2)}$). Next, we argue that the upper cone cannot have measure zero since it contains a random real. Finally, one uses this fact to isolate X as a path in a μ -r.e. tree. The last step is a generalized version of the result that if the Turing upper cone of a real has positive Lebesgue measure, then the real must be computable [5, 42]. Our presentation follows [36].

Proof of Lemma 2.13. Suppose $X \leq_{\text{T}} Y \oplus \mu$ via a Turing functional Φ and $Y \leq_{\text{T}} \mu^{(n-1)}$. Note that Y is Δ_2^0 relative to $\mu^{(n-2)}$. Let $Y(n, s)$ be a $\mu^{(n-2)}$ -recursive approximation of Y , i.e., $\lim_s Y(n, s) = Y(n)$. Given $i, s \in \omega$, put

$$U_{i,s} = \{X : \exists t > s (\Phi_t^{X \oplus \mu}(i) = Y(i, t))\}.$$

The set $U_{i,s}$ is $\Sigma_1^{0,\mu^{(n-2)}}$ uniformly in i, s and hence $P = \bigcap_{i,s} U_{i,s}$ is $\Pi_2^{0,\mu^{(n-2)}}$. Note that P is the upper cone of X under Φ ,

$$P = \{A : \Phi(A) = X\}.$$

P cannot have μ -measure 0: If it had then, since Borel probability measures on 2^ω are regular, for the sequence of open sets $(V_k)_{k \in \omega}$ given by $V_k = \bigcap_{\langle i,s \rangle \leq k} U_{i,s}$, we have $\mu V_k \searrow 0$. Since each V_k is $\Sigma_1^{0,\mu^{(n-2)}}$, $\mu^{(n-1)}$ can

decide whether $\mu V_k \leq 2^{-l}$ for given l . Hence, we can convert (V_k) into a (μ, n) -test. Since $\bigcap_k V_k = P$ and P contains Y , this contradicts the fact that Y is μ - n -random.

Hence pick r rational such that $\mu P > r > 0$, where r is rational. Define a tree T by letting

$$\sigma \in T \iff \mu\{\tau: \Phi(\tau \oplus \mu) \supseteq \sigma\} > r,$$

and closing under initial segments. T is r.e. in μ and X is an infinite path through T .

Since μ is a probability measure, any antichain in T contains at most $\lceil 1/r \rceil$ strings. Choose $\sigma = X \upharpoonright_n$ such that no $\tau \supseteq \sigma$ incompatible with X is in T . Such σ exists for otherwise we could find an antichain of more than $\lceil 1/r \rceil$ strings branching off X . To compute $X \upharpoonright_m$ from μ , it suffices to enumerate T above σ until a long enough extension shows up. \square

We will later need the following relativization of the previous lemma. The proof is similar.

Lemma 2.14. *Suppose μ is a continuous measure and Y is μ - $(k+n)$ -random, $k \geq 0, n \geq 2$. If $X \leq_{\text{T}} \mu^{(k+n-1)}$ and $X \leq_{\text{T}} Y \oplus \mu^{(k)}$, then $X \leq_{\text{T}} \mu^{(k)}$.*

One interpretation of Lemmas 2.13 and 2.14 is that μ -random reals are not helpful in computing (defining) reals arithmetic in μ . For example, if a real is properly Δ_n^0 relative to a measure μ , then it cannot be Δ_k^0 relative to $\mu \oplus Y$ where $k < n$ and Y is μ - $(n+1)$ -random.

In Section 4, we will also need a result similar to the previous lemmas regarding initial segments of linear orders, namely, that random reals are not helpful in the recognizing well-founded initial segments. The following lemma may appear technical at this point, but its importance will become clear towards the end of Section 4, in the proof of Theorem 4.53. We use it to separate, in a definable way, well-founded from non-well-founded models.

Lemma 2.15. *Let $j \geq 0$. Suppose μ is a continuous measure and \prec is a linear order on a subset of ω such that the relation \prec and the field of \prec are both recursive in $\mu^{(j)}$. Suppose further Y is $(j+5)$ -random relative to μ , and $I \subseteq \omega$ is the longest well-founded initial segment of \prec . If I is recursive in $(Y \oplus \mu)^{(j)}$, then I is recursive in $\mu^{(j+4)}$.*

Proof. Suppose $I \leq_{\text{T}} (Y \oplus \mu)^{(j)}$, Y is $(j+5)$ -random relative to μ , but $I \not\leq_{\text{T}} \mu^{(j+4)}$. By Lemma 2.11 (with $Z = \mu^{(j)}$, $n = 5$, and $X = I$), there is a continuous measure $\mu_I \leq_{\text{T}} \mu^{(j+2)}$ such that I is $(\mu_I, \mu^{(j+2)}, 3)$ -random.

For given $a \in \text{Field}(\prec)$, let $\mathcal{I}(a)$ be the set of all reals $A \subseteq \omega$ such that A is an initial segment of \prec , and all elements of A are bounded by a . $\mathcal{I}(a)$ is a $\Pi_1^0(\mu^{(j)})$ class. Let T_a be a tree recursive in $\mu^{(j)}$ such that $[T_a] = \mathcal{I}(a)$. Given $n \in \omega$, let $T_a \upharpoonright_n = \{\sigma \in T_a : |\sigma| = n\}$ be the n -th level of T_a . We have $\mathcal{I}(a) = \bigcap_n \llbracket T_a \upharpoonright_n \rrbracket$.

Now, if $a \in I$, then $\mathcal{I}(a)$ is countable (since in this case each element of $\mathcal{I}(a)$ is an initial segment of the well-founded part of \prec and there are at most countably many such initial segments). Since μ_I is continuous, it follows that $\mathcal{I}(a)$ has μ_I -measure zero.

If, on the other hand, $a \notin I$, then $I \in \mathcal{I}(a)$. Since I is $(\mu_I, \mu^{(j+2)}, 3)$ -random and $\mathcal{I}(a)$ is $\Pi_1^0(\mu^{(j)})$, $\mathcal{I}(a)$ does not have μ_I -measure zero: Otherwise we could recursively in $\mu^{(j+2)}$, compute a sequence (l_n) such that $\mu_I \llbracket T_a \upharpoonright_{l_n} \rrbracket \leq 2^{-n}$. This would be a $(\mu_I, \mu^{(j+2)}, 1)$ -test that covers I , but I is $(\mu_I, \mu^{(j+2)}, 3)$ -random.

We obtain the following characterization of I .

$$a \in I \iff \forall n \exists l (\mu_I \llbracket T_a \upharpoonright_l \rrbracket \leq 2^{-n})$$

Since $\mu_I \leq_{\text{T}} \mu^{(j+2)}$, the property on the right hand side is $\Pi_2^0(\mu^{(j+2)})$, hence I is recursive in $\mu^{(j+4)}$, contradicting our initial assumption. \square

We conclude this section by establishing that Turing jumps cannot be μ - n -random, $n \geq 2$, for any measure μ . While strictly speaking the following two results are not needed later, they are prototypical for a type of argument that will be important in Section 4, where we construct long sequences of reals with an internal definability hierarchy that are not random with respect to any continuous measure.

Proposition 2.16. *For any $k \geq 0$, if $X \equiv_{\text{T}} \emptyset^{(k)}$, then X is not 2-random with respect to any continuous measure.*

Proof. The case $k = 0$ is clear, so assume $k > 0$. Suppose $X \equiv_{\text{T}} \emptyset^{(k)}$ is μ -2-random for some μ . Then $\emptyset' \leq_{\text{T}} X$ and also $\emptyset' \leq_{\text{T}} \mu'$. It follows from Lemma 2.13 that \emptyset' is recursive in μ . Applying the same argument inductively to $\emptyset^{(i)}$, $i \leq k$, yields $\emptyset^{(i)} \leq_{\text{T}} \mu$, in particular $X \equiv_{\text{T}} \emptyset^{(k)} \leq_{\text{T}} \mu$, which is impossible if X is μ -2-random. \square

It may be helpful to picture the preceding argument as a “stair trainer machine”: Using the supposedly random X , each step, that is, Turing jump, “sinks down” to μ , and eventually, $X \leq_{\text{T}} \mu$, yielding a contradiction.

The non-randomness property of the jumps extends to infinite jumps, too.

Proposition 2.17. *If $X \equiv_T \emptyset^{(\omega)}$, then X is not 3-random with respect to a continuous measure.*

Proof. Assume for a contradiction that X is μ -3-random for continuous μ . By the inductive argument of the previous proof, $\emptyset^{(k)} \leq_T \mu$ for all $k \in \omega$. By a result of Enderton and Putnam [10], if Y is a \leq_T -upper bound for $\{\emptyset^{(k)} : k \in \omega\}$, then $\emptyset^{(\omega)} \leq_T Y''$. Therefore, $X \leq_T \mu''$, contradicting that X is μ -3-random. \square

3. THE COUNTABILITY THEOREM

In this section we will prove Theorem 1, which we restate here for convenience.

Theorem 1. *Let $n \in \omega$. Then the set*

$$\text{NCR}_n = \{X \in 2^\omega : X \text{ is not } n\text{-random for any continuous measure}\}$$

is countable.

As mentioned in the introduction, the case $n = 1$ was proved in [41]. The basic outline for the proof for $n > 1$ is as follows. By Borel-Turing determinacy [31], the set of degrees of reals X such that X is n -random for some continuous measure recursive in X contains an upper cone in the Turing degrees. The base of the cone is given by the winning strategy in a certain Borel game $\mathcal{G}(B)$. The constructive nature of Martin's proof of Borel determinacy yields that a winning strategy is contained in a countable level $L_{\beta_{n+4}}$ of the constructible hierarchy. We use a forcing notion due to Kumabe and Slaman (see [46]) to show that given $X \notin L_{\beta_{n+4}}$, there exists a forcing extension $L_{\beta_{n+4}}[\Phi]$ such that Φ is a real number and every real in $L_{\beta_{n+4}}[\Phi]$ is Turing reducible to X relative to Φ . In particular, $X \oplus \Phi$ is in the upper cone of random reals above the winning strategy for the game $\mathcal{G}(B)$ (in $L_{\beta_{n+4}}[\Phi]$). Finally, we argue that X itself is random with respect to a continuous measure.

3.1. About Borel determinacy. We will apply Martin's theorem about the determinacy of Borel games in the following special case.

Borel-Turing determinacy ([31, 32]). *If $A \subseteq 2^\omega$ is a Turing invariant Borel set, then either A or $2^\omega \setminus A$ contains a Turing cone.*

Martin's proof of Borel determinacy is explicit. It locates a winning strategy for a given Borel game within the constructible hierarchy.

Definition 3.1. Given $n \in \omega$, ZFC_n^- denotes the axiom of ZFC, where the power set axiom is replaced by the sentence

“There exist n -many iterates of the power set of ω ”.

Hence, in ZFC_0^- , for instance, we have the existence of the set of all natural numbers (since the Axiom of Infinity holds and ω is absolute), and various other subsets of ω as given by applications of separation or replacement, but we lack the guaranteed existence of the set of all such subsets.

Models of ZFC_n^- will play an important role throughout this paper. In particular, we are interested in models inside the constructible universe. As usual, L will denote the constructible universe, the limit of the cumulative hierarchy of sets obtained by iterating the power set operation restricted to definable subsets. For any ordinal α , L_α denotes the α -th level of the hierarchy. A key property of this hierarchy is that $|L_\alpha| = |\alpha|$ for any infinite ordinal α . For more background on L , see [21, Chapter 13] or [28, Chapters V,VI]. For an in-depth account, see [7].

Definition 3.2. Given $n \in \omega$, let β_n be the least ordinal such that

$$L_{\beta_n} \models \text{ZFC}_n^-.$$

By the Löwenheim-Skolem theorem and the Gödel condensation lemma, L_{β_n} , and hence β_n , is countable.

Lemma 3.3. *If $A \subseteq 2^\omega$ is Σ_n^0 , then the Borel game $\mathcal{G}(A)$ with winning set A has a winning strategy S in L_{β_n} .*

The proof given by Martin [33] is inductive. A key concept is the *unraveling of a game*. Simply speaking, a tree T over some set B unravels $\mathcal{G}(A)$ if there exists a continuous mapping $\pi: [T] \rightarrow 2^\omega$ such that $\pi^{-1}(A)$ is clopen in $[T]$, and there is a continuous correspondence between strategies on T and strategies on $2^{<\omega}$.

Martin first shows that Π_1^0 games can be unraveled. The argument is completely constructive, hence can be carried out in L . The unraveling tree T is given by the legal moves of some auxiliary game whose moves correspond to strategies in the original game on $2^{<\omega}$, that is, reals. To be able to collect all these legal moves requires the existence of the power set of ω .

The inductive step then shows how to unravel a given Σ_n^0 set A . Suppose $A = \bigcup A_i$, where each A_i is Π_{n-1}^0 . By induction hypothesis, each A_i can be unraveled by some T_i via some mapping π_i . Martin proves that the unravelings T_i can be combined into a single one, T_∞ , that unravels each

A_i via some π_∞ . Since each of the sets $\pi_\infty^{-1}(A_i)$ is clopen, their union $\bigcup \pi_\infty^{-1}(A_i) = \pi_\infty^{-1}(A)$ is open, and can in turn be unraveled by some T . Again, the proof is constructive. The last step in the construction (unraveling $\pi_\infty^{-1}(A)$) passes to a tree of higher type – its nodes correspond to strategies over T_∞ . Hence one more iterate of the power set of ω is introduced.

Therefore, Σ_n^0 determinacy is provable in ZFC_n^- , and Martin's proof constructs a winning strategy S in L_{β_n} , relative to L_{β_n} . The property of being a winning strategy for a given Borel game is $\mathbf{\Pi}_1^1$. By Mostowski's absoluteness theorem (see [21, Theorem 25.4], which applies because in a model of ZFC_n^- , $n \geq 1$, every well-founded tree contained in $\omega^{<\omega}$ has a rank function), this means that S actually wins on *all* plays, not just the ones in L_{β_n} .

In order to prove Theorem 1, we need a Posner-Robinson style theorem for reals not contained in L_{β_n} and an application of absoluteness. We state Lemma 3.4 only in the case that n is greater than zero. Under this restriction, we can avoid class forcing and reduce to standard facts about set forcing. The case n equals zero is not needed for our argument.

Lemma 3.4. *Suppose that n is a natural number greater than zero and X is a real number not in L_{β_n} . Then there exists a real $\Phi \subseteq \omega$ such that $L_{\beta_n}[\Phi]$ is a model of ZFC_n^- and every real in $L_{\beta_n}[\Phi]$ is Turing reducible to $X \oplus \Phi$.*

3.2. Kumabe-Slaman forcing. This subsection is devoted to proving Lemma 3.4. We derive Φ from a generic G which we construct by means of a notion of forcing due to Kumabe and Slaman. The forcing was an essential ingredient in the proof of the definability of the Turing jump by Shore and Slaman [46]. It allows for extending the Posner-Robinson Theorem to iterated applications of the Turing jump.

In the following, we use the conventions and vocabulary of Section 2.1.

Definition 3.5. Let \mathbb{P} be the following partial order.

- (1) The elements p of \mathbb{P} are pairs (Φ_p, \vec{Z}_p) in which Φ_p is a finite, use-monotone Turing functional and \vec{Z}_p is a finite set of subsets of ω . As usual, we identify subsets of ω with elements of 2^ω .
- (2) If p and q are elements of \mathbb{P} , then $p \geq q$ if and only if
 - (a) (i) $\Phi_p \subseteq \Phi_q$ and
 - (ii) for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length of σ_p ,
 - (b) $\vec{Z}_p \subseteq \vec{Z}_q$,

(c) for every x, y , and $Y \in \vec{Z}_p$, if $\Phi_q(x, Y) = y$ then $\Phi_p(x, Y) = y$.

In short, a stronger condition than p can add computations to Φ_p , provided that they are longer than any computation in Φ_p and that they do not apply to any element of \vec{Z}_p .

Let \mathbb{P}_n denote the partial order \mathbb{P} as defined in L_{β_n} . By standard arguments, we show that if $G \subseteq \mathbb{P}_n$ is a generic filter in the sense of L_{β_n} , then $L_{\beta_n}[G]$ is a model of ZFC_n^- . By inspection of \mathbb{P}_n , any such G naturally gives rise to a functional $\Phi_G = \bigcup\{\Phi_p : p \in G\}$. To prove Lemma 3.4, given X not in L_{β_n} , we will exhibit a particular G so that G is \mathbb{P}_n -generic over L_{β_n} and so that every element in $L_{\beta_n}[G]$ is computable from $X \oplus \Phi_G$.

Definition 3.6 (Definition III.3.3, [28]). Let \mathbb{P}^* be a partially ordered set. Then, $p, q \in \mathbb{P}^*$ are *compatible* iff they have a common extension. An *antichain* is a subset of \mathbb{P}^* whose elements are pairwise incompatible. \mathbb{P}^* has the *countable chain condition* (ccc) iff, in \mathbb{P}^* every antichain is countable.

Lemma 3.7. *Let n be a natural number greater than zero.*

- (1) \mathbb{P}_n is an element of L_{β_n} .
- (2) $L_{\beta_n} \models \mathbb{P}_n$ has the ccc.

Proof. Since $n \geq 1$, by definition of β_n , the power set of ω as defined in L_{β_n} is a set in L_{β_n} . Since \mathbb{P} is defined directly from the power set of ω , \mathbb{P}_n is a set in L_{β_n} .

If p and q are incompatible elements of \mathbb{P}_n , then Φ_p and Φ_q must be different. Since there are only countably many possibilities for Φ_p and Φ_q , any antichain in \mathbb{P}_n must be countable in L_{β_n} . \square

Definition 3.8. Let G be a subset of \mathbb{P}_n .

- (1) G is a *filter* on \mathbb{P}_n iff
 - (a) G is not empty.
 - (b) $\forall p, q \in G \exists r \in G [p \geq r \text{ and } q \geq r]$.
 - (c) $\forall p, q \in G [\text{if } p \geq q \text{ and } q \in G, \text{ then } p \in G]$.
- (2) G is \mathbb{P}_n -*generic* over L_{β_n} iff for all D such that $D \subseteq \mathbb{P}_n$ is dense and $D \in L_{\beta_n}$, $G \cap D$ is not empty. Here, D is dense iff for every $p \in \mathbb{P}_n$ there is a $q \in D$ such that $p \geq q$.

Lemma 3.9. *If G is \mathbb{P}_n -generic over L_{β_n} , then $L_{\beta_n}[G] \models \text{ZFC}_n^-$. Similarly, $L_{\beta_n}[\Phi_G]$ is a model of ZFC_n^- .*

Proof. By Lemma IV.2.26 of [28], it follows that if G is \mathbb{P}_n generic over L_{β_n} , then $L_{\beta_n}[G]$ satisfies the axioms of ZF^- except for possibly Replacement.

It remains only to observe that $L_{\beta_n}[G]$ satisfies Replacement, Choice and that there are n -many uncountable cardinals. The verification that $L_{\beta_n}[G]$ satisfies Replacement is the same as given in Theorem IV.2.27 of [28]. That $L_{\beta_n}[G]$ satisfies Choice follows from Replacement and the usual proof that the order of constructibility is a L_{β_n} -definable well-order of L_{β_n} applies relative to G . Finally, that $L_{\beta_n}[G]$ has the same uncountable cardinals as L_{β_n} does follows from \mathbb{P}_n 's having the ccc in L_{β_n} by the argument given in the proof of Theorem IV.3.4 of [28]. Although this theorem is stated for models of ZFC, its proof does not invoke the power set axiom.

Given that $L_{\beta_n}[G]$ has n many uncountable cardinals, we proceed to apply a variant of the Gödel Condensation Lemma relative to G to show that $L_{\beta_n}[G]$ has n -many iterates of the power set of ω .

The existence of the power set of ω is a special case. Let ω_1^* denote $\omega_1^{L_{\beta_n}[G]}$. We may view G as a subset of ω_1^* . We show that any subset of ω in $L_{\beta_n}[G]$ is an element of $L_{\omega_1^*}(G)$, the structure obtained by constructing relative to G as a predicate to height ω_1^* . Assume $x \in L_\gamma[G]$, $\gamma < \beta_n$, and $\gamma > \omega_1^*$. Let H be a countable elementary substructure of $L_\gamma[G]$ in $L_{\beta_n}[G]$ that includes x . Let $\pi : H \rightarrow \bar{H}$ be the transitive collapse of H . Note that π is the identity on all countable ordinals in H . Hence $\pi(G)$ is $G \cap \pi(\omega_1^*)$ and $\pi(x) = x$. By the Condensation Lemma, \bar{H} is isomorphic to some $L_{\gamma^*}[G \cap \pi(\omega_1^*)]$ and $\gamma^* < \omega_1^*$. It follows that $x \in L_{\gamma^*}[G \cap \pi(\omega_1^*)]$, and thus it is an element of $L_{\omega_1^*}(G)$.

The case for the higher order power sets is the standard Condensation Lemma argument.

Finally, $L_{\beta_n}[\Phi_G]$ is a model of ZFC_n^- because it is a definable inner model of $L_{\beta_n}[G]$, so it satisfies ZFC^- , and the Condensation Lemma applies, yielding the same number of iterates of the power set of ω . \square

Next, we show that every dense set in L_{β_n} can be met via an extension adding no computations along X . This is crucial for the construction in [46].

Lemma 3.10. *Let $D \in L_{\beta_n}$ be dense in \mathbb{P}_n and $X \in 2^\omega$, $X \notin L_{\beta_n}$. For any $p \in \mathbb{P}_n$, there exists a $q \leq p$ such that $q \in D$ and Φ_q does not add any new computation along X .*

Proof. Suppose $p = (\Phi_p, \vec{Z}_p)$ is in \mathbb{P}_n . We say a string τ is *essential* for (p, D) if, whenever $q < p$ and $q \in D$, there exists a triple $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ such that σ is compatible with τ . In other words, whenever one meets D by an extension of p , a computation relative to some string compatible with τ is added. Note that τ 's being essential for (p, D) is definable in L_{β_n} .

If τ is a binary sequence and τ_0 is an initial segment of τ , then any sequence σ which is compatible with τ is also compatible with τ_0 . Thus, being essential is closed under taking initial segments. So,

$$T(p, D) = \{ \tau : \tau \text{ essential for } (p, D) \}$$

is a binary tree in L_{β_n} .

Assume now for a contradiction that a q as postulated above does not exist. This means that for any $r \leq p$, either $r \notin D$ or Φ_r adds a computation along X . It follows that every initial segment $\tau \subset X$ is essential for (p, D) . Thus $T(p, D)$ is infinite. Since L_{β_n} satisfies König's Lemma (equivalently, compactness of the Cantor set), there exists a real $Y \in 2^\omega \cap L_{\beta_n}$ such that Y is an infinite path through $T(p, D)$.

Now consider the condition $p_1 = (\Phi_p, \vec{Z}_p \cup \{Y\})$. As $\Phi_q = \Phi_p$ and $Y \in L_{\beta_n}$, we trivially have $p_1 \leq p$ in \mathbb{P}_n . Since every initial segment of Y is essential for (p, D) , any extension of p in D must add a computation along Y . Since no extension of p_1 can add a computation along Y and every extension of p_1 is an extension of p , no extension of p_1 is in D . This contradicts the density of D . \square

The previous lemma also holds in the case $X \in L_{\beta_n}$, but we are only interested in the case $X \notin L_{\beta_n}$.

We can now finish the proof of Lemma 3.4. It is sufficient to construct a \mathbb{P}_n -filter G that is generic over L_{β_n} such that for every $A : \omega \rightarrow 2$ in $L_{\beta_n}[G]$ there is a k such that for all m , $\Phi_G^X((k, m)) = A(m)$. We fix countings of \mathbb{P}_n , of the set of terms τ in the forcing language for functions from ω to 2 in L_{β_n} and of the dense subsets D of \mathbb{P}_n in L_{β_n} . We let $(\tau_i : i \in \omega)$ and $(D_i : i \in \omega)$ denote the latter two countings. We proceed by recursion on s to define G . At stage s , we will define a condition $p_s = (\Phi_{p_s}, \vec{Z}_{p_s})$ and an integer k_s . We will ensure that for all i and m , $\Phi_G^X((k_i, m))$ will have the same value as the interpretation of $\tau_i(m)$ in L_{β_n} . We define p_0 to be the empty condition. Suppose that p_s and k_0, \dots, k_{s-1} are given. First, we define k_s to be the least integer k such that for all $(x, y, \sigma) \in \Phi_{p_s}$, x is not of the form (k, m) for any m . Next, let q be the least condition, according to our counting of \mathbb{P}_n , such that q extends p_s , $q \in D_s$, q decides the values of $\tau_i(j)$ for each $i \leq s$ and $j \leq s$, and $\Phi_{p_s}^X = \Phi_q^X$. By Lemma 3.10, there is such a q . For i and j less than or equal to s , let $m_{i,j}$ be the value decided for $\tau_i(j)$ by q . Since $X \notin L_{\beta_n}$ and \vec{Z}_q is finite, there is an ℓ such that for each element $Z \in \vec{Z}_q$, Z and X disagree on some number less than ℓ . By increasing ℓ if necessary, we

may further assume that ℓ is greater than the length of any σ mentioned in an element of Φ_q . Let

$$\Phi_{p_{s+1}} = \Phi_q \cup \{(k_i, m_{i,j}, X \upharpoonright \ell) : i, j \leq s \text{ and } \Phi_q^X((i, j)) \text{ is not defined}\}.$$

Let p_{s+1} be $(\Phi_{p_{s+1}}, \vec{Z}_q)$. By construction, p_{s+1} is an extension of q in \mathbb{P}_n . Finally, let G be the filter of conditions in \mathbb{P}_n which are extended by some p_s and let Φ_G be the union of the set of Φ_p such that $p \in G$.

We conclude by observing that the set Φ_G , viewed as a subset of ω , satisfies the two conclusions of Lemma 3.4. First, we argue that $L_{\beta_n}[\Phi_G]$ is model of ZFC_n^- . Note that the cardinals of $L_{\beta_n}[G]$ are the cardinals of L_{β_n} , and since $\Phi_G \in L_{\beta_n}[G]$ they are also the cardinals of $L_{\beta_n}[\Phi_G]$. By Gödel's proof of the GCH, applied to $L_{\beta_n}[\Phi_G]$, $L_{\beta_n}[\Phi_G]$ has n iterates of the power set of ω . That means $L_{\beta_n}[\Phi_G]$ satisfies ZFC_n^- .

Second, for each $Z : \omega \rightarrow 2$, if $Z \in L_{\beta_n}[\Phi_G]$ then there is an i such that Z is the denotation of τ_i in $L_{\beta_n}[G]$. By direct induction on the construction, for this i , for all m , $Z(m) = \Phi_G^X((k_i, m))$.

3.3. The proof of Theorem 1. We now put the pieces together to show that every real outside of $L_{\beta_{n+4}}$ is n -random with respect to a continuous probability measure. As $L_{\beta_{n+4}}$ is countable, this will complete the proof of Theorem 1.

Given $X \notin L_{\beta_{n+4}}$, choose Φ as in Lemma 3.4, that is, every real in $L_{\beta_{n+4}}[\Phi]$ is Turing reducible to $X \oplus \Phi$.

Consider the game with the following winning set

$$B^\Phi = \{W \in 2^\omega : \exists Z \exists R (W \oplus \Phi \equiv_T Z \oplus \Phi \oplus R \ \& \ R \text{ is } (Z \oplus \Phi, n+3)\text{-random})\}.$$

Clearly, B^Φ is Turing invariant. To see that it is Borel, note that $W \oplus \Phi$ is in B^Φ if and only if

$\exists e, d$ (e, d are indices of Turing functionals such that

$$\Phi_d(\Phi_e(W \oplus \Phi)) = W \oplus \Phi,$$

$$\Phi_e(W \oplus \Phi) \text{ is of the form } Z \oplus \Phi \oplus R,$$

and R is $(n+3)$ -random relative to $Z \oplus \Phi$).

This is a prima facie arithmetic definition of B^Φ . By counting quantifiers, we see that B^Φ is $\Sigma_{n+4}^0(\Phi)$. We also know that the Turing degrees of B^Φ are cofinal in the Turing degrees (since it is always possible to “add on” another random).

By Lemma 3.3, $L_{\beta_{n+4}}[\Phi]$ contains a winning strategy S for the relativized game $\mathcal{G}(B^\Phi)$. Since B^Φ is cofinal in the Turing degrees, by Borel-Turing determinacy S has to be a winning strategy for the player who wins if the result of the game is in B^Φ . If any strategy is played against a real that computes the strategy, the result of game is Turing equivalent to the real the strategy was played against.

Therefore, since $X \oplus \Phi$ computes S , B^Φ is Turing invariant, and $(X \oplus \Phi) \oplus \Phi$ is Turing equivalent to $X \oplus \Phi$, $X \oplus \Phi$ is in B^Φ . Let Z and R witness the condition that $X \oplus \Phi \in B^\Phi$. Since R is $(Z \oplus \Phi, n+3)$ -random, $X \oplus \Phi$ is not recursive in $(Z \oplus \Phi)'$. By Lemma 2.11, $X \oplus \Phi$ is $(\nu, (Z \oplus \Phi)'', n+1)$ -random for some continuous measure ν recursive in $(Z \oplus \Phi)''$.

Let $\tilde{\nu}$ be the measure on 2^ω given by

$$\tilde{\nu}(A) = \nu(\{W \oplus \Phi : W \in A\})$$

Since $X \oplus \Phi$ is ν -random, and it belongs to the $\Pi_1^0(\Phi)$ set

$$2^\omega \oplus \Phi = \{W \oplus \Phi : W \in 2^\omega\},$$

$\nu(2^\omega \oplus \Phi) > 0$, and therefore $\tilde{\nu}(2^\omega) > 0$. Furthermore, since ν is continuous, so is $\tilde{\nu}$.

For any $\sigma \in 2^{<\omega}$, $\tilde{\nu}[\sigma]$ is recursive in $(\nu \oplus \Phi)'$, uniformly in σ . It follows that the normalization $\tilde{\nu}_N$ of $\tilde{\nu}$

$$\tilde{\nu}_N = \frac{1}{\tilde{\nu}(2^\omega)} \tilde{\nu}$$

is recursive in $(\nu \oplus \Phi)'$, too. Further, for any $\sigma \in 2^{<\omega}$ and any rational $\varepsilon > 0$, we can compute, recursively in $(\nu \oplus \Phi)'$, an open cover U of

$$\sigma \oplus \Phi = \{W \oplus \Phi : W \in [\sigma]\}$$

so that

$$|\nu(\sigma \oplus \Phi) - \nu(U)| < \varepsilon,$$

uniformly in σ and ε .

Thus, for any $k \geq 1$, there is uniformly recursive in $(\nu \oplus \Phi)^{(k)}$ procedure converting a Martin-Löf test for $(\tilde{\nu}_N)$ relative to $(\nu \oplus \Phi)^{(k)}$ into a Martin-Löf test for ν relative to $(\nu \oplus \Phi)^{(k)}$. Since $X \oplus \Phi$ is $(\nu, (Z \oplus \Phi)'', n+1)$ -random and ν is recursive in $(Z \oplus \Phi)''$, $X \oplus \Phi$ is $(\nu, (\nu \oplus \Phi), n+1)$ -random. Therefore, X is $(\tilde{\nu}_N, (\nu \oplus \Phi)', n)$ -random. Finally, note that every $(\tilde{\nu}_N, (\nu \oplus \Phi)', n)$ -random real is $(\tilde{\nu}_N, n)$ -random.

We have shown that every real not contained in $L_{\beta_{n+4}}$ is n -random for a continuous measure. As $L_{\beta_{n+4}}$ is countable, this completes the proof of Theorem 1.

4. THE METAMATHEMATICS OF RANDOMNESS

In this section, we will show that the metamathematical ingredients used to prove the countability of NCR_n are necessary. More precisely, we will prove the Theorem 2, which we restate here for convenience.

Theorem 2. *There exists a computable function $G(n)$ such that for every $n \in \omega$,*

$$\text{ZFC}_n^- \not\vdash \text{“NCR}_{G(n)} \text{ is countable.”}$$

Before starting the proof, we outline its basic idea. For given n , we will show that in the model L_{β_n} of ZFC_n^- , $\text{NCR}_{G(n)}$ is not countable. To this end, we find a sequence (Y_α) of reals that satisfies

- (1) (Y_α) is cofinal in the Turing degrees of L_{β_n} , hence not countable in L_{β_n} ,
- (2) no Y_α is $G(n)$ -random for a continuous measure in L_{β_n} .

As we have seen in Propositions 2.16 and 2.17, iterating the Turing jump produces an increasing sequence of non-random reals. It makes sense therefore to look for a set-theoretic analogue of the jump hierarchy that is cofinal in the hierarchy of constructibility within the countable structure L_{β_n} . Each initial segment of this hierarchy has a canonical representation in L as a countable object. These are called *master codes* for the structures that they represent. They induce a well-ordered, increasing sequence under Turing reducibility, just like the Turing jump. The ‘Stair Trainer’-argument to come will proceed along this hierarchy, as it did along the jump hierarchy in Propositions 2.16 and 2.17.

We will show that none of these master codes can be (sufficiently) random with respect to a continuous measure μ . In the Stair Trainer method, we compare the sets recursive in a fixed jump of μ to the elements of the hierarchy on which we are implementing the method. There are new challenges to be met here that go beyond iterates of the Turing jump. For example, we can have representations which are not canonical representations and we can have representations of non-wellfounded versions of the constructible hierarchy. Most of the technical work we have to do is to organize the representations of structures that are definable from μ -to linearize them and to eliminate

the non-wellfounded ones. The application of the Stair Trainer method then comes by comparing the canonical representations to the organized μ -definable representations, and then showing that there is no first place where randomness can occur.

This section of the paper is laid out as follows. 4.1 and 4.2 review some basic facts of Jensen’s fine structure analysis of the constructible hierarchy. Readers familiar with fine structure theory can skip ahead to Subsection 4.3. In 4.3 and 4.4, we study representations (which we call ω -copies) of countable levels of the constructible hierarchy from a computability point of view. We show how we can computationally retrieve the information coded in an ω -copy, and how we can propagate representations of lower levels to obtain representations of higher levels. In 4.5, we extract an arithmetic property which captures much of the combinatorics of the fine structure. Representations with this weaker property will be called *pseudocopies*. In 4.6, we implement the organization of pseudocopies recursive in a given real as described above. In 4.7 and 4.8, we implement the Stair Trainer technique and complete the proof of Theorem 4.

4.1. Fine structure and Jensen’s J-hierarchy. Fine structure provides a level-by-level, quantifier-by-quantifier analysis of how new sets are generated in L . Jensen defines the new constructible hierarchy, the *J-hierarchy* $(J_\alpha)_{\alpha \in \text{Ord}}$ that has all the important properties of the L -hierarchy (in particular, $L = \bigcup_\alpha J_\alpha$). In addition to this, each level J_α has closure properties (such as under pairing functions) that L_α may be lacking. While it is not strictly necessary for this paper to work with J_α (we could work with $(L_{\omega\alpha})_{\alpha \in \text{Ord}}$), the J -hierarchy is the established framework for fine structure analysis, and we will adopt its basic concepts and terminology.

The sets J_α are obtained by closing under a scheme of *rudimentary functions*. In contrast to $L_{\alpha+1}$, $J_{\alpha+1}$ contains sets of rank up to $\omega(\alpha + 1)$, not just subsets of J_α , e.g. ordered pairs. The rudimentary functions are essentially a scheme of *primitive set recursion* [23].

For transitive X , $\text{rud}(X)$ denotes the smallest set Y that contains $X \cup \{X\}$ and is closed under rudimentary functions (rud closed). The inclusion of $\{X\}$ when taking the rudimentary closure guarantees that new sets are introduced even if X is closed under rudimentary functions.

The J -hierarchy is introduced as a cumulative hierarchy induced by the rud-operation:

$$\begin{aligned}
J_0 &= \emptyset \\
J_{\alpha+1} &= \text{rud}(J_\alpha) \\
J_\lambda &= \bigcup_{\alpha < \lambda} J_\alpha \quad \text{for } \lambda \text{ limit.}
\end{aligned}$$

A fine analysis of the rudimentary functions reveals that the rud-operation can be completed by iterating some or all of *nine basic rudimentary functions*.

Proposition 4.1 (Jensen [22]). *Every rudimentary function is a composition of the following nine functions:*

$$\begin{aligned}
F_0(x, y) &= \{x, y\}, \\
F_1(x, y) &= x \setminus y, \\
F_2(x, y) &= x \times y, \\
F_3(x, y) &= \{(u, z, v) : z \in x \wedge (u, v) \in y\}, \\
F_4(x, y) &= \{(u, v, z) : z \in x \wedge (u, v) \in y\}, \\
F_5(x, y) &= \bigcup x, \\
F_6(x, y) &= \text{dom}(x), \\
F_7(x, y) &= \in \cap (x \times x), \\
F_8(x, y) &= \{\{x(z)\} : z \in y\}.
\end{aligned}$$

The S -operator is defined as taking a one-step application of any of the basic functions,

$$(4.1) \quad S(X) = [X \cup \{X\}] \cup \left[\bigcup_{i=0}^8 F_i[X \cup \{X\}] \right].$$

For transitive X , it holds that [22, Corollary 1.10]

$$\text{rud}(X) = \bigcup_{n \in \omega} S^{(n)}(X).$$

The S -hierarchy is defined as the cumulative hierarchy induced by the S -operator and refines the J -hierarchy.

$$\begin{aligned}
S_0 &= \emptyset, \\
S_{\alpha+1} &= S(S_\alpha), \\
S_\lambda &= \bigcup_{\alpha < \lambda} S_\alpha \quad \text{for } \lambda \text{ limit.}
\end{aligned}$$

We obviously have

$$J_\alpha = \bigcup_{\beta < \omega\alpha} S_\beta = S_{\omega\alpha}.$$

We list a few basic properties of the sets J_α . For details and proofs, see [22] or [7].

- Each J_α is transitive and is a model of a sufficiently large fragment of set theory (more precisely, it is a model of KP-set theory without Σ_0 -collection).
- The hierarchy is *cumulative*, i.e., $\alpha \leq \beta$ implies $J_\alpha \subseteq J_\beta$.
- $\text{rank}(J_{\alpha+1}) = \text{rank}(J_\alpha) + \omega$. Each successor step adds ω new ordinals. $J_\alpha \cap \text{Ord} = \omega\alpha$, in particular, $J_1 = V_\omega$ and $J_1 \cap \text{Ord} = \omega$.
- $(J_\alpha)_{\alpha \in \text{Ord}}$ and $(L_\alpha)_{\alpha \in \text{Ord}}$ generate the same universe: $L = \bigcup_\alpha J_\alpha$. Moreover, $L_\alpha \subseteq J_\alpha \subseteq L_{\omega\alpha}$, and $J_\alpha = L_\alpha$ if and only if $\omega\alpha = \alpha$. Finally, $J_{\alpha+1} \cap \mathcal{P}(J_\alpha) = \mathcal{P}_{\text{DEF}}(J_\alpha)$, that is, $J_{\alpha+1}$ contains precisely those subsets of J_α that are first order definable over J_α .
- The Σ_n -satisfaction relation over J_α , $\models_{J_\alpha}^{\Sigma_n}$, is Σ_n -definable over J_α , uniformly in α .
- The mapping $\beta \mapsto J_\beta$ ($\beta < \alpha$) is uniformly Σ_1 -definable over any J_α , and hence so is the set $\{J_\beta : \beta < \alpha\}$.
- There is a Π_2 formula $\varphi_{V=L}$ such that for any transitive set M ,

$$M \models \varphi_{V=L} \Leftrightarrow \exists \alpha M = J_\alpha.$$

In particular, $\varphi_{V=L}$ says that for every set a there exists an ordinal α such that a is in the rud-closure of J_α .

The J -hierarchy shares all important metamathematical features with the L -hierarchy. We cite the two most important facts. The L -versions of the two propositions together constitute the core of Gödel's proof that GCH and AC hold in L .

Proposition 4.2. *There exists a Σ_1 -definable well-ordering $<_J$ of L and for any $\alpha > 1$, the restriction of $<_J$ to J_α is uniformly Σ_1 -definable over J_α .*

Proposition 4.3 (The condensation lemma for J). *For any α , if $X \preceq_{\Sigma_1} J_\alpha$, then there is an ordinal β and an isomorphism π between X and J_β . Both β and π are uniquely determined.*

For proofs of these results for the J -hierarchy again refer to [22] or [7].

4.2. Projecta and master codes. The definable well-ordering $<_J$ together with the definability of the satisfaction relation can be used to show that each J_α has *definable Skolem functions*, essentially by selecting the $<_J$ -least witness that satisfies an existential formula. The definable Skolem functions can in turn be used to define an indexing of J_α [22, Lemma 2.10].

Proposition 4.4 (Jensen [22]). *For each α , there exists a $\Sigma_1(J_\alpha)$ -definable surjection from $\omega\alpha$ onto J_α .*

While a simple cardinality argument yields that $|J_\alpha| = |\omega\alpha|$, Jensen's result shows that an $\omega\alpha$ -counting of J_α already exists in $J_{\alpha+1}$. The indexing is obtained by taking (essentially) the Skolem hull of $\omega\alpha$ under the canonical Σ_1 -Skolem function. The resulting set X is a Σ_1 -elementary substructure of J_α , hence by the condensation lemma is isomorphic to some J_β . The isomorphism taking X to J_β is the identity on all ordinals below $\omega\alpha$, and one can show that this in turn implies that the isomorphism must be the identity on X , i.e., $X = J_\beta$.

Boolos and Putnam [3] first observed that if a new real is defined in $L_{\alpha+1}$, i.e., if

$$\mathcal{P}(\omega) \cap (L_{\alpha+1} \setminus L_\alpha) \neq \emptyset,$$

then the strong absoluteness properties of L can be used to get a *definable* ω -counting of L_α (instead of just an α -counting as above). Because, if a new subset Z of ω is constructed in $L_{\alpha+1} \setminus L_\alpha$, one can take the Skolem hull of ω instead of $\omega\alpha$. The resulting $X \cong L_\beta$ is still equal to L_α , since the definition of the new real applies in the elementary substructure L_β . If $\beta < \alpha$, then this would contradict the fact that $Z \notin L_\alpha$.

Proposition 4.5 (Boolos and Putnam [3]). *If $\mathcal{P}(\omega) \cap (L_{\alpha+1} \setminus L_\alpha) \neq \emptyset$, then there exists a surjection $f : \omega \rightarrow L_\alpha$ in $L_{\alpha+1}$.*

Of course, at some stages no new reals are constructed. Boolos and Putnam [3] showed that the first such stage is precisely the ordinal β_0 , i.e., the least ordinal β such that $L_\beta \models \text{ZF}^-$. By Gödel's work, on the other hand, we know that no new real is constructed after stage ω_1^L .

Jensen [22] vastly extended these ideas into the framework of *projecta* and *master codes*, which form the core concepts of *fine structure theory*.

Definition 4.6. For natural numbers $n > 0$ and ordinals $\alpha > 0$, the Σ_n -*projectum* ρ_α^n is equal to the least $\gamma \leq \alpha$ such that $\mathcal{P}(\omega\gamma) \cap (\Sigma_n(J_\alpha) \setminus J_\alpha) \neq \emptyset$.

We put $\rho_\alpha^0 = \alpha$. Hence $1 \leq \rho_\alpha^n \leq \alpha$ for all n . As ρ_α^n is non-increasing in n , we can also define

$$\rho_\alpha = \min_n \rho_\alpha^n \quad \text{and} \quad n_\alpha = \min\{k: \rho_\alpha^k = \rho_\alpha\}.$$

Jensen [22, Theorem 3.2] proved that the projectum ρ_α^n is equal to the least $\delta \leq \alpha$ such that there exists a function f that is $\Sigma_n(J_\alpha)$ -definable over J_α such that $f(D) = J_\alpha$ for some $D \subseteq \omega\delta$, establishing the analogy with the Boolos-Putnam result. From this it follows that if $\rho_\alpha^n < \alpha$, it must be a cardinal in J_α , for all n .

Jensen gave another characterization of the projectum, which in fact he used as his original definition in [22]. Suppose $\langle M, \in \rangle$ is a set-theoretic structure. We can extend this structure by adding an additional relation $A \subset M$. If we do this, we would like the structure to satisfy some basic set theoretic closure properties. For instance, we would like our universe to satisfy the comprehension axiom with respect to the new relation, that is, whenever we pick an $x \in M$, the collection of elements in x that satisfy A should be in M . Such structures are called *amenable*.

Definition 4.7. Given $A \subseteq M$, the structure $\langle M, A \rangle$ is called *amenable*, if M is a transitive set and

$$\forall x \in M [x \cap A \in M].$$

Jensen [22, Theorem 3.2] showed that

$$\begin{aligned} \rho_\alpha^n &= \text{the largest ordinal } \gamma \leq \alpha \text{ such that} \\ &\langle J_\gamma, A \rangle \text{ is amenable for any } A \subseteq J_\gamma \text{ that is in } \Sigma_n(J_\alpha). \end{aligned}$$

This means the projectum ρ_α^n identifies the “stable” core of J_α with respect to Σ_n definability over J_α .

Being amenable with rud-closed domain can also be characterized via relative rud-closedness. This will be important later.

Definition 4.8 (Jensen [22]). A function f is *A-rud* if it can be obtained as a combination of the basis functions F_1, \dots, F_8 and the function

$$F_A(x, y) = x \cap A.$$

A structure $\langle M, A \rangle$ is *rud closed* if $f[M^n] \subseteq M$ for all *A-rud* functions f .

Proposition 4.9 (Jensen [22]). *A structure $\langle M, A \rangle$, $A \subseteq M$, is rud closed if and only if M is rud closed and $\langle M, A \rangle$ is amenable.*

The existence of a definable surjection between (a subset of) $\omega\rho_\alpha^n$ and $\Sigma_n(J_\alpha)$ allows for coding $\Sigma_n(J_\alpha)$ into its projectum. One way this can be implemented is via so-called *master codes*.

Definition 4.10. A Σ_n *master code* for J_α is a set $A \subseteq J_{\rho_\alpha^n}$ that is $\Sigma_n(J_\alpha)$, such that for any $m \geq 1$,

$$\Sigma_{n+m}(J_\alpha) \cap \mathcal{P}(J_{\rho_\alpha^n}) = \Sigma_m(\langle J_{\rho_\alpha^n}, A \rangle).$$

A Σ_n master code does two things:

- (1) It “accelerates” definitions of new subsets of $J_{\rho_\alpha^n}$ by n quantifiers.
- (2) It replaces parameters from J_α in the definition of these new sets by parameters from $J_{\rho_\alpha^n}$ (and the use of A as an “oracle”).

The existence of master codes follows rather easily from the existence of a $\Sigma_n(J_\alpha)$ -mapping from $\omega\rho_\alpha^n$ onto J_α . However, for $n > 1$, this mapping is *not uniform*. Jensen exhibited a uniform, canonical way to define master codes, by iterating Σ_1 -definability.

Put

$$A_\alpha^0 = \emptyset, \quad p_\alpha^0 = \emptyset.$$

Assuming that A_α^n is a Σ_n master code, it is not hard to see that every set $x \in J_{\rho_\alpha^n}$ is Σ_1 -definable over $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ with parameters from $J_{\rho_\alpha^{n+1}}$ and one parameter from $J_{\rho_\alpha^n}$ (used to define a surjection from $\omega\rho_\alpha^{n+1}$ onto $J_{\rho_\alpha^n}$). Hence we can put

$$p_\alpha^{n+1} = \text{the } <_J\text{-least } p \in J_{\rho_\alpha^n} \text{ such that every } u \in J_{\rho_\alpha^n} \text{ is } \Sigma_1 \text{ definable} \\ \text{over } \langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \text{ with parameters from } J_{\rho_\alpha^{n+1}} \cup \{p\} .$$

The p_α^n are called the *standard parameters*.

Using p_α^{n+1} , we can code the Σ_1 elementary diagram of the structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ into a set A_α^{n+1} :

$$A_\alpha^{n+1} := \{(i, x) : i \in \omega \wedge x \in J_{\rho_\alpha^{n+1}} \wedge \langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \varphi_i^{(2)}(x, p_\alpha^{n+1})\},$$

where $(\varphi_i^{(k)})$ is a standard Gödel numbering of all Σ_1 formulas with k free variables. It is not hard to verify that A_α^{n+1} is a Σ_{n+1} master code for J_α . Furthermore, the structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ is amenable for each $\alpha > 1$, $n \geq 0$. We will call the structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ the *standard Σ_n J -structure* for J_α .

Definition 4.11. We denote the standard J -structure over J_α at the ‘ultimate’ projectum n_α by

$$\langle J_{\rho_\alpha}, A_\alpha \rangle := \langle J_{\rho_\alpha^{n_\alpha}}, A_\alpha^{n_\alpha} \rangle.$$

One consequence of the A_α^n being master codes is that we can obtain the sequence of projecta of an ordinal by iterating taking Σ_1 -projecta relative to $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$. Given an amenable structure $\langle J_\alpha, A \rangle$, the Σ_n -projectum $\rho_{\alpha, A}^n$ of $\langle J_\alpha, A \rangle$ is defined to be the largest ordinal $\rho \leq \alpha$ such that $\langle J_\rho, B \rangle$ is amenable for any $B \subseteq J_\rho$ that is in $\Sigma_n(\langle J_\alpha, A \rangle)$.

Proposition 4.12 (Jensen [22]). *For $\alpha > 1, n \geq 0$,*

$$\rho_\alpha^{n+1} = \rho_{\rho_\alpha^n, A_\alpha^n}^1.$$

In particular, the standard Σ_{n+1} J -structure for $J_\alpha = \langle J_\alpha, \emptyset \rangle$ is the standard Σ_1 J -structure for $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$.

4.3. ω -Copies of J -structures. We later want to apply the recursion theoretic techniques of Section 2 to countable J -structures. We therefore have to code them as subsets of ω . If the projectum ρ_α^n is equal to 1, all set-theoretic information about the J -structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ is contained in the master code A_α^n , which is simply a real, and hence lends itself directly to recursion theoretic analysis. Starting with the work by Boolos and Putnam [3], this has been studied in a number of papers (e.g. [24], [19]).

In this subsection we give a recursion theoretic analysis of the internal workings of a countable presentation of a J -structure.

Definition 4.13. Let $X \subseteq \omega$. The *relational structure* induced by X is $\langle F_X, E_X \rangle$, where

$$xE_Xy \Leftrightarrow \langle x, y \rangle \in X$$

and

$$F_X = \text{Field}(E_X) = \{x : \exists y (xE_Xy \text{ or } yE_Xx)\}.$$

The idea is that a number x is a code for the set whose elements are the sets coded by the numbers y with yE_Xx ,

$$\text{Set}_X(x) = \{y : yE_Xx\}.$$

For ease of notation, $\text{Set}_X(x)$ will also denote the structure obtained by restricting E_X to $\text{Set}_X(x)$.

The relational structure $\langle F_X, E_X \rangle$ is *extensional* if

$$\forall x, y \in F_X [(\forall z zE_Xx \Leftrightarrow zE_Xy) \Rightarrow x = y],$$

that is

$$\forall x, y \in F_X (x \neq y \Rightarrow \text{Set}_X(x) \neq \text{Set}_X(y)).$$

Mostowski's Collapsing Theorem states that if $\langle F_X, E_X \rangle$ is extensional and well-founded, it is isomorphic to a unique structure (M, \in) , where M is a transitive set. In this sense we can speak of a countable set theoretic structure *coded* by X . If $\varphi(v_1, \dots, v_n)$ is a formula in the language of the set theory, we can interpret it over $\langle F_X, E_X \rangle$ and write

$$X \models \varphi[a_1, \dots, a_n]$$

for $\langle F_X, E_X \rangle \models \varphi[a_1, \dots, a_n]$ with $a_i \in F_X$.

J -structures have an additional set A , and we capture this on the coding side via pairs $\langle X, M \rangle$, where $M \subseteq F_X$. Semantically, A and M are seen as interpreting a predicate added to the language. This way we can consider the satisfaction relation $\langle X, M \rangle \models \varphi$, where φ is a set-theoretic formula with an additional unary predicate.

We are particularly interested in relational structures that code countable standard J -structures. The following is a generalization of the definition due to Boolos and Putnam [3]

Definition 4.14. (1) An ω -copy of a countable, extensional, set-theoretic structure $\langle S, A \rangle$, $A \subseteq S$, is a pair $\langle X, M \rangle$ of subsets of ω such that X codes the structure $\langle F_X, E_X \rangle$ in the sense of Definition 4.13, and such that there exists a bijection $\pi : S \rightarrow F_X$ such that

$$(4.2) \quad \forall x, y \in S [x \in y \iff \pi(x)E_X\pi(y)],$$

and

$$(4.3) \quad M = \{ \pi(x) : x \in A \}.$$

(2) When π is an isomorphism between a J -structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ and the structure coded by $\langle X, M \rangle$ in the sense of (1), we say that $\langle X, M \rangle$ is an ω -copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ via π .

The definition thus means an ω -copy $\langle X, M \rangle$ of $\langle S, A \rangle$ is isomorphic to $\langle S, A \rangle$ when seen as structures over the language of set theory.

If $A = \emptyset$, then necessarily $M = \emptyset$, and in this case we say X is an ω -copy of S .

We will now consider ω -copies of standard J -structures.

If $\rho_\alpha^n = 1$, we have $J_{\rho_\alpha^n} = L_\omega = V_\omega$, i.e., the *hereditarily finite sets*. In this case, we obtain an ω -copy by fixing a recursive bijection π_ω from J_1 to ω , by which we mean that $\Delta_1(J_1)$ relations are mapped to recursive relations on ω uniformly. Such a mapping can be found, for instance, in [1].

We let $\langle x, y \rangle \in X$, i.e., $x E_X y$, if and only if $\pi_\omega^{-1}(x) \in \pi_\omega^{-1}(y)$ and $x \in M$ if and only if $\pi_\omega^{-1}(x) \in A_\alpha^{n+1}$. Then $\langle X, M \rangle$ is an ω -copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π_ω .

Definition 4.15. When $\rho_\alpha^n = 1$, the *canonical copy* of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ is the ω -copy defined above.

We will now show that from a canonical copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$, we can extract ω -copies of all $\langle J_{\rho_\alpha^i}, A_\alpha^i \rangle$, $i \leq n$, in an effective and uniform way.

By choice of p_α^{n+1} , for every $u \in J_{\rho_\alpha^n}$, there exists a Σ_1 -formula $\psi(v_0, v_1, v_2)$ and $x \in J_{\rho_\alpha^{n+1}}$ such that u is the only solution over $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ to $\psi(v_0, x, p_\alpha^{n+1})$.

Recall that $(\varphi_i^{(k)})$ is a standard Gödel numbering of all Σ_1 formulas with k free variables. We can assume that every formula $\varphi_i^{(k)}$ has a lead existential quantifier (by adding dummy variables if necessary). Let $\theta_i^{(k+1)}$ be such that $\varphi_i^{(k)}$ is $\exists v_0 \theta_i^{(k+1)}$.

Definition 4.16. A pair (i, x) , $i \in \omega$, $x \in J_{\rho_\alpha^{n+1}}$ is an *n-code* if there exists a $u \in J_{\rho_\alpha^n}$ such that u is the unique solution to

$$\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \theta_i^{(3)}(v_0, x, p_\alpha^{n+1}).$$

We can check the property of being an *n-code* using Σ_1 formulas: (i, x) is an *n-code* if and only if

$$(4.4) \quad \langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \exists v_0 \theta_i^{(3)}(v_0, x, p_\alpha^{n+1})$$

and

$$(4.5) \quad \langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \not\models \exists v_0, v_1 (\theta_i^{(3)}(v_0, x, p_\alpha^{n+1}) \wedge \theta_i^{(3)}(v_1, x, p_\alpha^{n+1}) \wedge v_0 \neq v_1).$$

This means a standard code has the information necessary to sort out *n-codes* among its elements. Relative to a canonical copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$, it is decidable whether some number is the image of an *n-code*, due to the effective way we translate between finite sets and their codes.

If $\langle X, M \rangle$ is an arbitrary ω -copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π , and $(i, x) \in J_{\rho_\alpha^{n+1}}$ is an *n-code*, then we call $\pi((i, x))$ a π -*n-code*. Being able to decide relative to $\langle X, M \rangle$ whether a number is a π -*n-code* of a pair hinges on knowledge of the following two functions

- (i) the mapping $n_X : n \mapsto \pi(n)$ ($n \in \omega$),
- (ii) the mapping $h_X : (\pi(x), \pi(y)) \mapsto \pi((x, y))$.

These mappings may not be computable relative to $\langle X, M \rangle$, but they are definable, as follows.

If $\alpha > 1$, then $\omega \in J_\alpha$, and an ω -copy of any such J_α must contain a witness for ω . In this case, we can recover n_X recursively in X' .

Lemma 4.17. *If X is an ω -copy of J_α , $\alpha \geq 1$, then the function n_X is uniformly computable in X' .*

Proof. We define n_X by recursion. $n_X(0)$ is the unique element of X that represents the empty set in X , i.e. $\forall x \neg xE_X n_X(0)$.

Similarly, $n_X(i+1)$ is the unique element of X such that

$$\forall a (aE_X n_X(i+1) \rightarrow a = n_X(i) \vee aE_X n_X(i)).$$

These properties are Π_1^0 uniformly in X , hence the function n_X is recursive in X' . \square

This argument applies more generally as follows.

Lemma 4.18. *If $\langle X, M \rangle$ is an ω -copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$, and $\rho_\alpha^n = 1$, then $\langle X, M \rangle'$ uniformly computes the canonical copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$.*

Proof. We want to decide whether a natural number a is in the image of A_α^n under π_ω . Given a , we first compute the diagram of the transitive closure of the finite set represented by a under π_ω . For any reasonable coding π_ω this will be recursive, but in the worst case it will be recursive in \emptyset' .

Next, we take the diagram and search inside X for the natural number b representing this set. This is a finite recursion that can be performed recursively in X' .

Then $a \in \pi_\omega[A_\alpha^n]$ if and only if $b \in M$. Therefore, $\pi_\omega[A_\alpha^n]$ is recursive in $\langle X, M \rangle'$. \square

We can also recover the function h_X arithmetically in X .

Lemma 4.19. *If X is an ω -copy of J_α , then the function h_X is uniformly computable in X' .*

Proof. We have

$$\begin{aligned} h_X(\pi(x), \pi(y)) = b &\Leftrightarrow \exists c, d [\forall z (zE_X c \Leftrightarrow z = \pi(x)) \\ &\quad \wedge \forall z (zE_X d \Leftrightarrow z = \pi(x) \vee z = \pi(y)) \\ &\quad \wedge \forall z (zE_X b \Leftrightarrow z = c \vee z = d)] \end{aligned}$$

It follows that h_X is a $\Sigma_2^0(X)$ function and as usual, a $\Sigma_2^0(X)$ function is $\Delta_2^0(X)$. \square

Definition 4.20. Suppose $\langle X, M \rangle$ is an ω -copy via π of a rud closed structure $\langle J, A \rangle$. We say $\langle X, M \rangle$ is *effective* if the function n_X is recursive in $X \oplus M$ and the function h_X is partial recursive in $X \oplus M$ and recursive in $X \oplus M$ on its domain $F_X \times F_X$.

Lemma 4.21. *If $\rho_\alpha^n = 1$, the canonical copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ is effective.*

Proof. The mapping π_ω^{-1} satisfies the conditions required in Definition 4.20 naturally. \square

Lemma 4.22. *If $\langle X, M \rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π , then the mapping*

$$\pi((u, v)) \mapsto (\pi(u), \pi(v)),$$

where $u, v \in J_{\rho_\alpha^{n+1}}$, is (partial) recursive in $X \oplus M$. The mapping

$$\pi((i, u)) \mapsto (i, \pi(u)),$$

where $i \in \omega$, is also (partial) recursive in $X \oplus M$.

Proof. The first mapping can be computed by inverting h_X (which must be one-to-one), the second mapping by additionally inverting n_X . \square

Lemma 4.23. *If $\langle X, M \rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π , then it is decidable in $X \oplus M$ whether a number $y \in \omega$ is a π - n -code.*

Proof. Suppose $y \in M$ (if not, it cannot be a π - n -code). Then $y = \pi((i, x))$ for some $(i, x) \in A_\alpha^{n+1}$. Since the copy is effective, we have

$$\pi((i, x)) = h_X(\pi(i), \pi(x)),$$

and by Lemma 4.22 we can find i and $\pi(x)$ recursively in $X \oplus M$.

Since $(\varphi_i^{(2)})$ is a standard Gödel numbering of the Σ_1 formulas with two free variables, there exist recursive functions g_1, g_2 such that $\varphi_{g_1(i)}^{(2)}$ and $\varphi_{g_2(i)}^{(2)}$ are Σ_1 formulas equivalent (over $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$) to the formulas in (4.4) and (4.5), respectively. Then (i, x) is a n -code if and only if $(g_1(i), x) \in A_\alpha^{n+1}$ and $(g_2(i), x) \notin A_\alpha^{n+1}$.

The latter two conditions are equivalent to

$$h_X(\pi(g_1(i)), \pi(x)) \in M \text{ and } h_X(\pi(g_2(i)), \pi(x)) \notin M,$$

which, since $\langle X, M \rangle$ is an effective copy, is recursive in $X \oplus M$. \square

Two n -codes (i_0, x_0) and (i_1, x_1) represent the same set $u \in J_{\rho_\alpha^n}$ if u is the unique solution to

$$\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \theta_{i_0}^{(3)}(v_0, x_0, p_\alpha^{n+1}) \text{ and } \langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \theta_{i_1}^{(3)}(v_0, x_1, p_\alpha^{n+1}).$$

A similar property can be defined for π - n -codes.

Lemma 4.24. *If $\langle X, M \rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π , then it is decidable in $X \oplus M$ whether two numbers are π - n -codes of the same set.*

Proof. (i_0, x_0) and (i_1, x_1) represent different sets if and only if

$$\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \exists v_0, v_1, v_2 [\theta_{i_0}^{(3)}(v_0, x_0, p_\alpha^{n+1}) \wedge \theta_{i_1}^{(3)}(v_1, x_1, p_\alpha^{n+1}) \wedge (v_2 \in v_0 \wedge v_2 \notin v_1) \vee (v_2 \notin v_0 \wedge v_2 \in v_1)].$$

Let $g_3(i_0, i_1)$ be a Gödel number for the Σ_1 formula

$$\psi(x, p_\alpha^{n+1}) \equiv \exists v_0, v_1 [\theta_{i_0}^{(3)}(v_0, (x)_0, p_\alpha^{n+1}) \wedge \theta_{i_1}^{(3)}(v_1, (x)_1, p_\alpha^{n+1}) \wedge v_0 \neq v_1]$$

Then (i_0, x_0) and (i_1, x_1) represent different sets if and only if

$$(g_3(i_0, i_1), (x_0, x_1)) \in A_\alpha^{n+1},$$

which in turn holds if and only if

$$h_X(\pi(g_3(i_0, i_1)), h_X(\pi(x_0), \pi(x_1))) \in M.$$

Since g_3 is computable, it follows from Lemma 4.22 that it is decidable in $X \oplus M$ whether two numbers are π - n -codes and whether they represent the same set. \square

Lemma 4.25. *If $\langle X, M \rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ via π, n_X, h_X , it computes an ω -copy $\langle Y, N \rangle$ of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$. Furthermore, the computation is uniform, and h_Y and n_Y can be computed uniformly from $X \oplus M$.*

Proof. By Lemmas 4.23 and 4.24, the set

$$U = \{y \in \omega : \exists u \in J_{\rho_\alpha^n} (y \text{ is the } <_\omega\text{-least } \pi\text{-}n\text{-code for } u)\}$$

is recursive in $X \oplus M$.

Let σ be the mapping

$$\sigma : u \in J_{\rho_\alpha^n} \mapsto \text{the unique } \pi\text{-}n\text{-code of } u \text{ in } U,$$

and put

$$Y = \{\langle \sigma(x), \sigma(y) \rangle : x \in y \in J_{\rho_\alpha^n}\}, \quad N = \{\sigma(x) : x \in A_\alpha^n\}.$$

Then $\langle Y, N \rangle$ is clearly an ω -copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$. To show that it is recursive in $X \oplus M$, we note that for $u, w \in J_{\rho_\alpha^n}$, if (i, x) is an n -code for u and (j, y) is an n -code for w ,

(4.6)

$$u \in w \Leftrightarrow \langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \exists v_0, v_1 (\theta_i^{(3)}(v_0, x, p_\alpha^{n+1}) \wedge \theta_j^{(3)}(v_1, y, p_\alpha^{n+1}) \wedge v_0 \in v_1).$$

Moreover,

$$(4.7) \quad u \in A_\alpha^n \Leftrightarrow \langle J_{\rho_\alpha^n}, A_\alpha^n \rangle \models \exists v_0 (\theta_i^{(3)}(v_0, x, p_\alpha^{n+1}) \wedge v_0 \in A_\alpha^n).$$

There are recursive functions g_4, g_5 that output Gödel numbers for Σ_1 formulas equivalent to the ones in (4.6) and (4.7), respectively. Given two numbers $a, b \in U$, we can use Lemma 4.22 to find (i, a_0) and (j, b_0) such that $a = h_X(\pi(i), \pi(a_0))$, $b = h_X(\pi(j), \pi(b_0))$. Then

$$a E_Y b \Leftrightarrow h_X(\pi(g_4(i, j)), h_X(a_0, b_0)) \in M.$$

Likewise,

$$a \in N \Leftrightarrow h_X(\pi(g_5(i)), \pi(a_0)) \in M.$$

To see that the functions h_Y and n_Y are uniformly recursive in $X \oplus M$ note that we can

- (i) given $i \in \omega$, effectively compute the Gödel number of a Σ_1 formula that is satisfied by u if and only if u is the natural number i ,
- (ii) given n -codes (i, x) , (j, y) for elements u, w in $J_{\rho_\alpha^n}$, compute a Gödel number for a Σ_1 formula whose only solution is (u, w) .

□

By iterating the procedure described above, we obtain the following.

Corollary 4.26. *If $\langle X, M \rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$, then it computes ω -copies of*

$$\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle, \langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1} \rangle, \dots, \text{ and } \langle J_{\rho_\alpha^0}, A_\alpha^0 \rangle = \langle J_\alpha, \emptyset \rangle = J_\alpha.$$

If a copy is not effective, we can use Lemma 4.19 to decode the predecessor J -structures.

Corollary 4.27. *If $\langle X, M \rangle$ is an ω -copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$, then $(X \oplus M)^{(2)}$ computes ω -copies of*

$$\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle, \langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1} \rangle, \dots, \text{ and } \langle J_{\rho_\alpha^0}, A_\alpha^0 \rangle = \langle J_\alpha, \emptyset \rangle = J_\alpha.$$

Lemma 4.28. *If X is an ω -copy of J_α , then X computes an ω -copy of $\langle J_{\rho_\beta^n}, A_\beta^n \rangle$, for all $n \in \omega$, $\beta < \alpha$.*

Proof. Both $J_{\rho_\beta^n}$ and A_β^n are elements of J_α . Let π be the isomorphism between J_α and X , and let $x_\beta, a_\beta^n \in F_X$ be such that

$$x_\beta = \pi(J_{\rho_\beta^n}), \quad a_\beta^n = \pi(A_\beta^n).$$

Then $\langle \text{Set}_X(x_\beta), \text{Set}_X(a_\beta^n) \rangle$, wherein $\text{Set}_X(x_\beta)$ is regarded as a structure and $\text{Set}_X(a_\beta^n)$ is regarded as a set. This is an ω -copy of $\langle J_{\rho_\beta^n}, A_\beta^n \rangle$, clearly recursive in X . \square

A similar argument yields an analogous fact for the S -operator.

Lemma 4.29. *If X is an ω -copy of J_α , then X computes an ω -copy of $S^{(n)}(J_\beta)$, for all $n \in \omega$, $\beta < \alpha$.*

4.4. Defining ω -copies. In the previous section we saw how to effectively extract information from ω -copies of J -structures. Next, we describe how ω -copies of new J -structures can be defined from ω -copies of given J -structures.

The J -hierarchy has two types of operations that we need to capture: defining new sets using the S -operator, and taking projecta and defining standard codes. We will analyze both operations from an arithmetic perspective.

An arithmetic analogue of the S -operator. The S -operator is defined by application of a finite number of explicit functions. This makes it possible to devise an arithmetic analogue, which we denote by \bar{S} , and which is the subject of the following lemma.

Lemma 4.30. *There exists an arithmetic function $\bar{S}(X) = Y$ such that, if X is an ω -copy of a transitive set U , $\bar{S}(X)$ is an ω -copy of the transitive closure of $S(U)$. Further, X is coded into a reserved column of ω , that is,*

$$\langle x, y \rangle \in X \Leftrightarrow \langle 2^x, 2^y \rangle \in \bar{S}(X),$$

and 3 represents the element $\{U\}$ in $\bar{S}(X)$.

Proof. The elements of $S(U)$ are obtained by single applications of the functions F_0, \dots, F_8 . Thus each element of $S(U)$ is the denotation of a term consisting of one of the functions and finitely many elements of $U \cup \{U\}$. We denote the set of terms by $T_S(U)$. Arithmetically in X , we can define an ω -copy $\bar{S}(X)$ of $S(U)$ from the natural X -recursive copy $T_S(X)$ of $T_S(U)$. Membership of the set denoted by one term in the set denoted by another term or equality between the sets denoted by terms is arithmetic in X , since these are defined by quantification over X . The same applies for elements of the transitive closure of the thus coded structure. The additional uniformity condition on the coding of X does not change the calculation. \square

We can subject the \bar{S} -operator to an analysis similar to that of the jump operator by Enderton and Putnam [10].

Lemma 4.31. *There exists an arithmetic predicate $Q(n, X, Y)$ such that if A is an ω -copy, $Q(n, A, Y)$ holds if and only if $Y = \overline{S}^{(n)}(A)$.*

Proof. Suppose the relation $\overline{S}(X) = Y$ is definable by a Π_k^0 formula. Given $Y \subset \omega$, let

$$Y_{[2]} = \{\langle a, b \rangle : \langle 2^a, 2^b \rangle \in Y\}.$$

For each $n \geq 1$, the relation $\overline{S}^{(n)}(X) = Y$ is definable via the formula

$$\psi_n \equiv Y = \overline{S}(Y_{[2]}) \wedge Y_{[2]} = \overline{S}((Y_{[2]})_{[2]}) \wedge \dots \wedge (Y)_{[2]}^{(n-1)} = \overline{S}(X),$$

where $(\cdot)_{[2]}^{(m)}$ denotes the m -th iterate of the $(\cdot)_{[2]}$ -operator. Each formula ψ_n is also Π_k^0 , and ψ_n can be found recursively and uniformly in n . Let $P(n, X, Y)$ be a universal Π_k^0 predicate, that is, for every Π_k^0 predicate $R(X, Y)$ there exists an $n \in \omega$ such that

$$P(n, X, Y) \Leftrightarrow R(X, Y),$$

and n can be found recursively from a formula defining R . Let $i(n)$ be the index with respect to P of the Π_k^0 predicate defined by ψ_n . The predicate

$$Q(n, X, Y) :\Leftrightarrow P(i(n), X, Y)$$

has the property claimed in the statement of the Lemma. \square

Lemma 4.32. *If Z is such that $Z \geq_T \overline{S}^{(n)}(X)$ for all n , then $\bigoplus_n \overline{S}^{(n)}(X)$ is uniformly arithmetically definable from Z .*

Proof. Define the predicate $\overline{Q}(n, e)$ as

$$\overline{Q}(n, e) :\Leftrightarrow \Phi_e^Z \text{ is total and } Q(n, X, \Phi_e^Z).$$

\overline{Q} is an arithmetic predicate relative to Z since $Z \geq_T \overline{S}(X) \geq_T X$. To decide whether $a \in \overline{S}^{(n)}(X)$, find, arithmetically in Z , the least e such that $\overline{Q}(n, e)$ and compute $\Phi_e^Z(a)$. \square

Corollary 4.33. *If X is an ω -copy of J_α and $Z \geq_T \overline{S}^{(n)}(X)$ for all n , then Z uniformly arithmetically defines an ω -copy of $J_{\alpha+1}$.*

Proof. We can use $\bigoplus_n \overline{S}^{(n)}(X)$ to define a copy of $J_{\alpha+1}$ by ‘stacking’ the elements of $\overline{S}^{(n+1)}(X)$ coded with base 3 and higher at the next ‘available’ prime column. Essentially this means that instead of moving $\overline{S}^{(n)}(X)$ into the column given by powers of 2, we leave it unchanged and add new elements for $\overline{S}^{(n+1)}(X)$ starting at the smallest available prime column. \square

An arithmetic version of the standard code. To define an arithmetic copy of the Σ_n -standard code for J_α , we can simply interpret the set theoretic definitions as formulas of arithmetic. More precisely, suppose P is a definable predicate over a J -structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$, and $\langle X, M \rangle$ is an ω -copy via π . Since the structure $\langle F_X, E_X, M \rangle$ is isomorphic to $\langle J_{\rho_\alpha^n}, \in, A_\alpha^n \rangle$, we can use the same formula that defines P over $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ and obtain a definition of $\pi[P]$ arithmetic in $\langle X, M \rangle$. The problem, however, is that a bounded quantifier in set theory will not necessarily correspond to a bounded quantifier in arithmetic. This means the transfer of complexities between the Lévy-hierarchy and the arithmetical hierarchy may not result in uniform bounds.

However, we will use only a fixed, finite number of set-theoretic definitions. Most importantly, we use the uniform definability of the satisfaction relation \models over transitive, rud closed structures.

Proposition 4.34 (Jensen [22], Corollary 1.13). *For $n \geq 1$, the satisfaction relation $\models_{\langle M, A \rangle}^{\Sigma_n}$ is uniformly $\Sigma_n(\langle M, A \rangle)$ for transitive, rud closed structures $\langle M, A \rangle$.*

Corollary 4.35. *Suppose $\langle X, M \rangle$ is an ω -copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$. Then there exists an ω -copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1} \rangle$ uniformly arithmetically definable in $\langle X, M \rangle$.*

Proof. The projectum ρ_α^{n+1} and the standard parameter are uniformly first-order definable over $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ (uniformly in n). Then the standard code A_α^{n+1} is uniformly defined over the same structure using the parameter ρ_α^{n+1} . These uniform first-order definitions become uniform arithmetic definitions in ω -copies, which yields the corollary. \square

4.5. Recognizing J-structures. Our goal is to show that there exists a recursive function G such that, for each n , no element of the sequence of canonical copies of J -structures with projectum equal to one in L_{β_n} can be $G(n)$ -random with respect to a continuous measure. In the proof of this result (Theorem 4.53), we need to consider the initial segment of ω -copies computable in (some fixed jump of) μ .

The problem is that we cannot arithmetically define the set of ω -copies of structures J_α . We can, however, define a set of “pseudocopies”, subsets of ω that behave in most respects like ω -copies of actual J_α , but that may code structures that are not well-founded. We can, arithmetically in the pseudocopy, require that its version of the natural numbers (if any) is isomorphic to ω , that is to say that it is an ω -model.

By comparing the structures coded by these pseudocopies, we can also linearly order a subset of the latter (up to isomorphism), depending on whether a coded structure embeds into another. This ordering will be developed in Section 4.6.

To define what it means to be a pseudocopy, we have to formulate “pseudo”-versions of the properties that characterize the J_α 's. These properties will stand in for the use of the Condensation Lemma, which we lack in non-wellfounded structures. One such property is rudimentary closure. A first-order approximation φ_{Rud} is the assertion that the formulas which specify the graphs of F_0, \dots, F_8 from Proposition 4.1 define total functions.

We also need to require the presence of a linearly-ordered, internal J -hierarchy inside a pseudocopy. By Proposition 4.2, for any β , the sequence of J_α ($\alpha < \beta$) is uniformly Σ_1 -definable over J_β . Let φ be the formula defining this collection. For an ω -copy X , we define

$$\vec{J}^X = \{z: X \models \varphi(z)\},$$

the J -structure inside X .

We will also apply some aspects of the fine structure of the J -hierarchy by requiring that pseudocopies exhibit the same features: Elements of the J -hierarchy satisfy $V = L$ and that there exists a Σ_1 -map from the ordinals of the structure onto the whole structure.

Lemma 4.36. *If X is an ω -copy that satisfies φ_{Rud} and is an ω -model, then the satisfaction predicate \models^{Σ_1} for Σ_1 -formulas is Σ_1 -definable in $\langle F_X, E_X \rangle$.*

Proof. The proof follows the argument given by Jensen for transitive, rud-closed sets [22, Lemma 1.12, Corollary 1.13]. The argument does not use the full well-foundedness of the \in -relation, but rather works with ω -models. \square

Definition 4.37. A set $X \subseteq \omega$ is a *pseudocopy* if the following hold.

- (1) The relation E_X is non-empty and extensional.
- (2) X codes an ω -model: if in the structure coded by X there is an upper bound on the codes of finite ordinals in X , then there is a code for a least upper bound in X .
- (3) The structure $\langle F_X, E_X \rangle$ satisfies φ_{Rud} .
- (4) The structure $\langle F_X, E_X \rangle$ satisfies
 - (a) $\varphi_{V=L}$,
 - (b) the ordinals in the structure are linearly ordered by E_X and no two distinct ordinals are isomorphic,

- (c) there is a Σ_1 -map from the ordinals of $\langle F_X, E_X \rangle$ onto F_X and for every $a \in F_x$ such that

$$\langle F_X, E_X \rangle \models \text{“}a \text{ is an infinite cardinal”},$$

it holds that

$$\langle F_X, E_X \rangle \models \text{“there is a bijection between } a \text{ and } a \times a \text{”}.$$

- (d) the elements represented by \vec{J}^X are transitive and linearly ordered by the \in -relation.
- (e) if $\varphi_{<}$ is the Σ_1 -formula that defines $<_J$ as in Proposition 4.2, this formula $\varphi_{<}$ defines a linear ordering in the structure $\langle F_X, E_X \rangle$ such that
- (i) every Σ_3 -definable property has a least element with respect to this linear ordering,
 - (ii) for every $z \in \vec{J}^X$, the ordering defined by $\varphi_{<}$ inside z is an initial segment of the ordering in $\langle F_X, E_X \rangle$.

Moreover, (a)-(e) hold for every set induced by some $z \in \vec{J}^X$. (As X codes an extensional structure, and z is transitive in X , this condition is equivalent to saying that X satisfies the five conditions hold with quantifiers bounded by z .)

Every ω -copy of a J_α is a pseudocopy. Below we use various properties of pseudocopies that are true of each J_α and are arithmetically definable for ω -copies. For the sake of efficiency, the reader may assume that these properties are part of the definition of a pseudocopy.

Lemma 4.38. *If X is a pseudocopy and $z \in \vec{J}^X$, then $\text{Set}_X(z)$ is a pseudocopy.*

Proof. (1) holds since, by item (4)(d), if $z \in \vec{J}^x$, then z represents a transitive set in X . Being extensional is absolute to transitive elements of X and E_X is extensional.

For (2), note that X and $\text{Set}_X(z)$ have the same natural numbers.

Item (3) holds for $\text{Set}_X(z)$ since inside X , z is rud-closed, and the graphs of the functions F_0, \dots, F_8 are absolute between X and $\text{Set}_X(z)$.

For (4), first note, by Lemma 4.36, item (c) is a first-order property. $\text{Set}_X(z)$ itself satisfies (a)-(e) specified in (4) because it is in \vec{J}^X . Now suppose $w \in \vec{J}^{\text{Set}_X(z)}$. Membership in \vec{J} is determined by a (uniformly) Σ_1 -formula and these formulas are upward absolute, so $w \in \vec{J}^X$. Since (4) holds for X , $\text{Set}_X(w)$ satisfies (a)-(e). Since $\text{Set}_{\text{Set}_X(z)}(w) = \text{Set}_X(w)$,

the condition that (a)-(e) hold in $\text{Set}_X(w)$ is a formula in which all the quantifiers are bounded by $\text{Set}_X(w)$, and such formulas are absolute between the structures coded by X and by z within X , $\text{Set}_X(w)$ satisfies (a)-(e), as required. \square

Lemma 4.39. *There exists an arithmetic formula $\varphi_{\text{PC}}(X)$ such that $\varphi_{\text{PC}}(X)$ holds for a real X if and only if X is a pseudocopy. Moreover, if $\langle F_X, E_X \rangle$ is well-founded, then it is an ω -copy of a countable J_β , $\beta > 1$.*

Proof. Given Lemma 4.38 and item (2), items (1), (3), and (4) are first-order properties and hence arithmetic relative to X .

With respect to (2), we can define ω using the usual Σ_0 set theoretic formula (the least infinite ordinal). In transitive, rud-closed sets, $\varphi_\omega(x)$ holds if and only if $x = \omega$. Interpreting φ_ω over $\langle F_X, E_X \rangle$, we obtain an arithmetic in X property. We require a pseudocopy $\langle F_X, E_X \rangle$ to satisfy

$$\exists x \varphi_\omega(x).$$

This x will be unique and define ω with respect to $\langle F_X, E_X \rangle$. Let us denote this unique number by ω_X .

Given ω_X , we can also recover the mapping $i \mapsto n_X(i)$ as in Lemma 4.17. As the definition of ω_X is uniform, we obtain that $i \mapsto n_X(i)$ is uniformly arithmetic in X . (2) holds exactly when this map from ω to ω_X is a surjection, which is again uniformly arithmetic relative to X .

The conjunction of the arithmetic formula that characterizes (2) with the arithmetic analogues of the first-order formulas characterizing (1), (3), and (4) yields the arithmetic formula φ_{PC} .

By Mostowski's Collapsing Theorem, if X satisfies (1) and E_X is well-founded, then $\langle F_X, E_X \rangle$ is isomorphic to a transitive set structure $\langle S, \in \rangle$, and by (3) S will be rud-closed. Finally, any rud-closed set that satisfies $\varphi_{V=L}$ is a J_α . \square

4.6. Comparing pseudocopies. If two pseudocopies X and Y define well-founded structures, they are ω -copies of sets J_α and J_β , respectively. Since $\alpha < \beta$ implies $J_\alpha \in J_\beta$, it follows that one structure must embed into the other as an initial segment.

We want to find an arithmetic formula that compares two pseudocopies in this respect. The problem is that, in general, the isomorphism relation between countable structures need not be arithmetic. In our case, however,

we can make use of the special set-theoretic structure present in the pseudocopies, by comparing the subsets of the cardinals present, to show that the isomorphism relation is arithmetic.

The complexity of the arithmetic operations involved in these comparisons will depend on the number of cardinals present in a pseudocopy.

Let us introduce the following notation. Recall that β_N denotes the least ordinal such that $L_{\beta_N} \models \mathbf{ZF}_N^-$. For any ordinal α , let

$$(4.8) \quad P_\alpha = \max\{n: \mathcal{P}^{(n)}(\omega) \text{ exists in } J_\alpha\},$$

if this maximum exists. We first note that for all $\alpha < \beta_N$, $P_\alpha \leq N$. This is because, if P_α were greater than or equal to $n + 1$ and β were the $(n + 1)$ st cardinal in L_α , then L_β would satisfy \mathbf{ZF}_N^- , hence $\alpha > \beta_N$. Hence P_α is defined and uniformly bounded by N for all $\alpha < \beta_N$.

Using the predicate φ_ω , we can formalize the (non-)existence of power sets of ω for pseudocopies. For any $k \in \omega$, there exists a formula defining the predicate $y = \mathcal{P}^{(k)}(\omega)$.

Definition 4.40. A pseudocopy X is an n -pseudocopy if it satisfies the uniformly arithmetic in X predicate

$$\exists y(y = \mathcal{P}^{(n)}(\omega)) \wedge \forall z(z \neq \mathcal{P}^{(n+1)}(\omega)).$$

We now use the fact that pseudocopies are ω -models. Using the power sets of ω in each pseudocopy, we can check whether two pseudocopies have the same reals, sets of reals, etc.

First, we can check whether every real in X has an analogue in Y :

$$\forall u (X \models u \subseteq \omega \rightarrow \exists v (Y \models v \subseteq \omega \wedge \forall i (n_X(i)E_X u \leftrightarrow n_Y(i)E_Y v))).$$

By extensionality, such a v , if it exists, is unique. We can therefore define the mapping $f_0^{X,Y}(u) = v$ which maps the representation of a real in $\langle F_X, E_X \rangle$ to its representation in $\langle F_Y, E_Y \rangle$. We can similarly check whether every real in Y has an analogue in X . This gives rise to a function $f_0^{Y,X}$. Let $\varphi_{\text{comp}}^{(0)}(X, Y)$ be the arithmetic formula asserting that X and Y code the same subsets of ω .

We can continue this comparison through the iterates of the power set of ω . This will yield arithmetic formulas $\varphi_{\text{comp}}^{(n)}(X, Y)$ with the following property:

If X and Y are pseudocopies in which $\mathcal{P}^{(n)}(\omega)$ exists, then $\varphi_{\text{comp}}^{(n)}(X, Y)$ holds if and only if X and Y have (representations of) the same subsets of $\mathcal{P}^{(i)}(\omega)$, for all $0 \leq i \leq n$.

Given two n -pseudocopies, the above formulas allow for an arithmetic definition of isomorphic pseudocopies.

Lemma 4.41. *For given n and for any two n -pseudocopies X, Y that code well-founded structures $\langle F_X, E_X \rangle$ and $\langle F_Y, E_Y \rangle$, respectively, if $\varphi_{\text{comp}}^{(n)}(X, Y)$, then X and Y code the same J_α .*

Proof. Assume for a contradiction X and Y are not isomorphic. Since they are well-founded pseudocopies, there must exist countable α, β such that $\langle F_X, E_X \rangle \cong (J_\alpha, \in)$ and $\langle F_Y, E_Y \rangle \cong (J_\beta, \in)$. Without loss of generality, $\alpha < \beta$. Since $\langle F_X, E_X \rangle$ and $\langle F_Y, E_Y \rangle$ code the same subsets of $\mathcal{P}^{(n)}(\omega)$ no new subset of $\mathcal{P}^{(n)}(\omega)$ is constructed between α and β . But this implies $\mathcal{P}^{(n+1)}(\omega)$ exists at $\alpha + 1$, which is an immediate contradiction if $\beta = \alpha + 1$. If $\beta > \alpha + 1$, since $\mathcal{P}^{(n+1)}(\omega)$ does not exist in J_β , a new subset of $\mathcal{P}^{(n)}(\omega)$ must be constructed between $\alpha + 1$ and β , contradiction. \square

We will consider the comparison between ill-founded structures later.

Corollary 4.42. *For given n , if X and Y are well-founded n -pseudocopies and $\varphi_{\text{comp}}^{(n)}(X, Y)$, then the isomorphism from the structure coded by X to the structure coded by Y is arithmetic in (X, Y) . Furthermore, the formula which defines this isomorphism depends only on n .*

Proof. Under the given hypothesis, there exists an α such that both X and Y are isomorphic to J_α .

Since $\varphi_{\text{comp}}^{(n)}(X, Y)$ holds, we can arithmetically in the pair (X, Y) match the power sets of ω up to n , as follows. First, observe that by Lemma 4.17, there is an order-isomorphism between the natural numbers in X and in Y . By the proof of the GCH, there are bijections in J_α between the k -th uncountable cardinal in J_α and the power sets of its predecessor cardinal in J_α . We can use these bijections to inductively find order isomorphisms between the cardinals in X and the cardinals in Y . Finally, this yields an order isomorphism between the greatest cardinals in X and Y , which is arithmetically definable in (X, Y) .

Next we show how to define a mapping between ordinals of these structures. Let γ be an ordinal in X . Since ω_n is the biggest cardinal in X , there exists a mapping from ω_n onto γ in X . Let f_X be the $<_J$ -least such mapping. The

function f_X induces a relation on ω_n in X . Since $<_J$ is uniformly $\Sigma_1(J_\alpha)$, arithmetically in (X, Y) we can find the $<_J$ -least function f_Y in Y which induces the same relation on ω_n . The image of ω_n under f_Y represents the ordinal γ in Y . The result of this process is the desired mapping.

By the proof of Proposition 4.4 given by Jensen, there is a $\Sigma_1(J_\alpha)$ -definable map from ω_α , that is the set of ordinals in J_α , onto J_α . The definition of this map involves a parameter p from J_α such that J_α is the Σ_1 Skolem hull of p and the ordinals of J_α . Arithmetically in (X, Y) , we can find the X -version and the Y -version of p and the map from finite sequences of ordinals in J_α corresponding to forming the Skolem hull. From these we get the isomorphism between X and Y . Note that the definition of the isomorphism is uniformly arithmetic in the pair (X, Y) and depends only on the number of iterates of the power set of ω present. \square

We can use the transfer function of Corollary 4.42 to translate also between copies of $S^{(n)}(J_\alpha)$.

Corollary 4.43. *For every N , there exists a number d_N , which can be computed uniformly from N , such that the following holds. Suppose X is an ω -copy of some J_α with $P_\alpha \leq N$. Suppose further that Z is an ω -copy of $S^{(n)}(J_\alpha)$, for some $n \in \omega$. Then $\bar{S}^{(n)}(X)$ is recursive in $(X \oplus Z)^{(d_N)}$.*

Proof. There is a k as follows. For a given ω -copy A of a transitive set a such that a is in the model coded by Z , say represented by parameter b , and a given isomorphism f between A and the encoding given by b in Z , if $S(a)$ is coded in Z then uniformly recursively in $A \oplus f \oplus Z^{(k)}$ there is an isomorphism between $\bar{S}(A)$ and b .

We obtain the isomorphism as follows. We start with the ω -copy B of the set a as encoded by b inside Z . Using the notation of the proof of Lemma 4.30, we form the term set copy $T_S(B)$ of $T_S(a)$. Arithmetically in Z , we can define maps g_1 from $T_S(B)$ to $\bar{S}(B)$ and g_2 from $\bar{S}(B)$ to Z 's copy of $S(a)$. Recursively in $f \oplus g_1$, we can obtain an isomorphism between $\bar{S}(A)$ and $\bar{S}(B)$: the isomorphism between the term sets is recursive, and g_1 provides the necessary information about identification of terms in \bar{S} . Finally, application of g_2 provides the isomorphism between $\bar{S}(A)$ and Z 's copy of $S(a)$.

Now we turn to the proof of the corollary. Since Z encodes $S^{(n)}(J_\alpha)$, it also encodes J_α . Uniformly arithmetically in $Z \oplus X$, we can find the element

in Z that encodes J_α and the isomorphism f between X and Z 's ω -copy of J_α .

By the above, recursively in $X \oplus f \oplus Z^{(k)}$, we can find an isomorphism, f_1 , between $\bar{S}(X)$ and Z 's ω -copy of $S(J_\alpha)$. By iterating the process, we obtain, recursively in $X \oplus f \oplus Z^{(k)}$, an isomorphism between $\bar{S}^{(n)}(X)$ and Z . Since f is arithmetic in $X \oplus Z$, the corollary follows. \square

It is worth highlighting that in the previous corollary we obtain a fixed arithmetic bound relative to Z which is independent of n .

For fixed $N \in \omega$, let

$$\mathcal{PC}_N = \{X : X \text{ is an } n\text{-pseudocopy for some } n \leq N \\ \text{and for every } Y \in \vec{J}^X, Y \text{ is a } k\text{-pseudocopy for some } k \leq N\}$$

This is an arithmetic set of reals.

Working inside \mathcal{PC}_N , we can also use φ_{comp} to arithmetically define a pre-order \prec on pseudocopies. The idea is that $X \prec Y$ if X embeds its structure into Y .

Lemma 4.44. (1) *Suppose $X \in \mathcal{PC}_N$ then the structure $\langle F_X, E_X \rangle$ has no non-trivial automorphism.*

(2) *If for some $z_1, z_2 \in \vec{J}^X$, $\text{Set}_X(z_1)$ is isomorphic to $\text{Set}_X(z_2)$, then $z_1 = z_2$.*

Proof. (1) We first argue there is a definable parameter and a mapping, which is Σ_1 relative to this parameter, from the subsets of the greatest cardinal ω_n^X , where $n \leq N$, onto $\langle F_X, E_X \rangle$. By item (4)(c) of Definition 4.37 there is a Σ_1 -definable (relative to some parameter) mapping from the ordinals of X onto X . By item (4)(e) among the parameters which can be used to define such a mapping there is a φ_{\prec} -least one (which thus makes the parameter definable). Since this parameter must be fixed by any automorphism, to prove (1) it is sufficient to find a mapping from subsets of the greatest cardinal onto the ordinals.

Given $a \subseteq \omega_n^X$, using the bijection between ω_n^X and $\omega_n^X \times \omega_n^X$ (guaranteed by item (4)(c) in Definition 4.37), construe a as a binary relation on a subset of ω_n^X . Map a to ordinal γ if there is an isomorphism between this relation and γ . Otherwise, map a to 0. A failure of this mapping's being well-defined would yield an existentially definable isomorphism between two distinct ordinals, contradicting item (4)(b) in Definition 4.37. Moreover, the mapping is onto because for every ordinal there is an injection of it into ω_n^X .

Now, by induction on $j \leq n$, an automorphism of the structure $\langle F_X, E_X \rangle$ has to fix every subset of ω_j^X . The case $j = 0$ follows from the fact that pseudocopies are ω -models. Hence the only automorphism of $\langle F_X, E_X \rangle$ is the identity.

(2) Assume $z_1, z_2 \in \vec{J}^X$. By item (4)(d) in the definition of pseudocopies, without loss of generality, $\text{Set}_X(z_1) \subseteq \text{Set}_X(z_2)$. By the argument for (1), an isomorphism from $\text{Set}_X(z_1)$ to $\text{Set}_X(z_2)$ has to be the identity on all subsets of the greatest cardinal of $\text{Set}_X(z_1)$, and hence the identity on all of $\text{Set}_X(z_1)$. Since an isomorphism is surjective, $z_1 = z_2$. \square

Lemma 4.45. *Suppose X and Y are in \mathcal{PC}_N . If X and Y code isomorphic structures ($X \cong Y$), the isomorphism is unique and uniformly arithmetically definable from the pair (X, Y) .*

Proof. Assume X and Y code isomorphic structures. Two different isomorphisms between the structures would yield a non-trivial automorphism of each, contradicting Lemma 4.44.

Let N_0 be such that there are N_0 -many infinite cardinals in the structure coded by X and Y . By induction on $n \leq N_0$ the isomorphism restricted to $\mathcal{P}^{(n)}(\omega)$ is arithmetic in $X \oplus Y$. The base of the induction is provided by the fact that both X and Y are ω -models. Inside each structure, there is a map $\mathcal{P}^{(N_0)}(\omega)$ to its greatest cardinal, κ , by which the isomorphism restricted to these greatest cardinals is arithmetic in $X \oplus Y$. Consequently, the isomorphism restricted to subsets the greatest cardinals, respectively, is arithmetic in $X \oplus Y$. Since every ordinal in each structure determined by a subset of the greatest cardinal, the isomorphism between the ordinals in each structure is arithmetic in $X \oplus Y$. Finally, to obtain an arithmetic isomorphism between X and Y , we use the existence of a Σ_1 -map from the ordinals to the whole structure, as given by item (4)(c) in Definition 4.37.

By the previous paragraph, there is a k , depending on N , by taking into account all the $N_0 \leq N$, such that the isomorphism between the structures code by X and Y , respectively, is recursive in $(X \oplus Y)^{(k)}$. Finding an index for this isomorphism is uniformly computable in $(X \oplus Y)^{(k+2)}$. \square

Corollary 4.46. *Suppose X and Y are in \mathcal{PC}_N . Whether $X \cong Y$ is a uniformly arithmetic property of the pair (X, Y) .*

Definition 4.47. We define

$$X \prec_N Y \quad :\Leftrightarrow X, Y \in \mathcal{PC}_N \wedge \exists z (z \in \vec{J}^Y \wedge X \cong \text{Set}_Y(z)).$$

By Corollary 4.46, this is an arithmetic property of the pair (X, Y) . We let $X \preceq_N Y$ if $X \prec_N Y$ or $X \cong Y$. Clearly, \preceq_N is reflexive. Since the property of being an element of the J -hierarchy is determined by a Σ_1 -formula, and these properties are upwards absolute, \preceq_N is transitive. Therefore, \preceq_N defines a partial order on \mathcal{PC}_N .

If both X and Y are well-founded pseudocopies in \mathcal{PC}_N , we have either $X \prec_N Y$ or $X \cong Y$ or $Y \prec_N X$, that is, “true” pseudocopies (i.e., those that code a J_α) are linearly pre-ordered by \preceq_N . Hence comparability can only fail if (at least) one of the pseudocopies is not well-founded.

Provided with a countable subset of \mathcal{PC}_N , such as all the elements of \mathcal{PC}_N recursive in a real Z , we will want to arithmetically define a subset that is linearly ordered by \preceq_N by excluding some ill-founded pseudocopies.

Definition 4.48. Given a real Z , let $\mathcal{PC}_N(Z)$ be the intersection of \mathcal{PC}_N with the set of reals computable in Z .

Lemma 4.49. *For every natural number N there is an arithmetic predicate such that for every real Z , the predicate defines a set of reals $\mathcal{PC}_N^*(Z) \subseteq \mathcal{PC}_N(Z)$ with the following properties:*

- (1) *For every $X \in \mathcal{PC}_N^*(Z)$ and $z \in F_X$, if $z \in \vec{J}^X$, then $\text{Set}_X(z) \in \mathcal{PC}_N^*(Z)$.*
- (2) *\preceq_N is a total preorder on $\mathcal{PC}_N^*(Z)$.*
- (3) *If $X \in \mathcal{PC}_N(Z)$ is well-founded, then $X \in \mathcal{PC}_N^*(Z)$.*

Proof. To start, note that if $z \in \vec{J}^X$ and $X \in \mathcal{PC}_N(Z)$, then $\text{Set}_X(z)$ is recursive in Z and then by applying Lemma 4.38, $\text{Set}_X(z) \in \mathcal{PC}_N(Z)$.

We show that if X, Y are \preceq_N -incomparable, at least one of two must be ill-founded. Further, this identification of ill-foundedness is uniformly arithmetic in the pair (X, Y) .

Suppose X, Y are \preceq_N -incomparable. Consider the predicate

$$\text{Iso}(x, y, X, Y) : \Leftrightarrow x \in \vec{J}^X \wedge y \in \vec{J}^Y \wedge \text{Set}_X(x) \cong \text{Set}_Y(y).$$

It yields an arithmetic partial function from \vec{J}^X to \vec{J}^Y . It is single-valued by Lemma 4.44. Denote the domain of this function by D_X and the range by R_Y . Both D_X and R_Y are linearly ordered by item (4)(d) of Definition 4.37.

Now we apply the incomparability of X and Y to show that there must be at least one instance of ill-foundedness in X or in Y .

Case 1: D_X is cofinal in X , R_Y is cofinal in Y .

This means, by the definition of the function for which D_X and R_Y are domain and range, respectively, the complete internal J -hierarchies of

X and Y , respectively, are pairwise isomorphic. Furthermore, these isomorphisms are compatible by Lemma 4.44. Their union hence exhibits an isomorphism between the structure coded by X and the structure coded by Y , contradicting \preceq_N -incomparability.

Case 2: D_X is cofinal in X , R_Y is bounded in Y .

Since $X \not\cong \text{Set}_Y(z)$ for any $z \in \bar{J}^Y$, Y must omit $\bigcup_{z \in R_Y} \text{Set}_Y(z)$ and hence the J -hierarchy in Y is ill-founded. The case when D_X is bounded and R_Y is cofinal is analogous.

Case 3: Both D_X, R_Y are bounded in X, Y , respectively.

In this case, D_X and R_Y are cuts in \bar{J}^X and \bar{J}^Y , respectively. If these cuts were principal in both structures, it would contradict the definition of D_X and R_Y by adding a new element to each set, as follows. In the limit case, reason as in Case 1: the union of the $\text{Set}_X(x)$, $x \in D_x$, as evaluated in X , maps to the union of the $\text{Set}_Y(y)$, $y \in R_Y$, as evaluated in Y . In the successor case, given an isomorphism between $\text{Set}_X(x)$ and $\text{Set}_Y(y)$, because X and Y code ω -models, there also exists an isomorphism between $S(\text{Set}_X(x))$, as evaluated in X , and $S(\text{Set}_Y(y))$, as evaluated in Y .

Therefore, at least one of the two cuts is not principal, thereby exhibiting an instance of non-wellfoundedness.

By inspection of the proof, the case distinction and the identification of the ill-founded structure is uniformly arithmetic in the pair (X, Y) .

We obtain the arithmetic set $\mathcal{PC}_N^*(Z)$ by considering all incomparable pairs in $\mathcal{PC}_N(Z)$ and discarding all elements of $\mathcal{PC}_N(Z)$ such that they or some element of their J -hierarchy are shown to be ill-founded by the above analysis. This ensures property (1). Since all pairs are being considered, $\mathcal{PC}_N^*(Z)$ is linearly pre-ordered by \preceq_N , ie. property (2) holds. For property (3) of the lemma, note that any element removed from $\mathcal{PC}_N(Z)$ in this process is ill-founded. \square

Taking limits of ω -copies. We can use the ordering \preceq_N to construct limits of ω -copies. This will be needed in the proof of Theorem 4.53.

Lemma 4.50. *For every N , there exists a number e_N , which can be computed uniformly from N , such that the following holds. Suppose $\bar{X} = \{X_i : i \in \omega\}$ is a family of well-founded pseudocopies from \mathcal{PC}_N , in other words, each X_i codes a countable J_{α_i} in which there are at most N uncountable cardinals.*

Let γ be the supremum of the α_i . Then there exists an ω -copy of J_γ recursive in $\overline{X}^{(\epsilon_N)}$.

Proof. We first note that if X_i and X_j code J_{α_i} and J_{α_j} , respectively, and $\alpha_i < \alpha_j$, then the embedding of J_{α_i} as coded by X_i into J_{α_j} as coded by X_j is uniformly arithmetic in \overline{X} . See Lemma 4.45. Thus, the directed system consisting of the structures coded by the X_i 's and the maps between them is arithmetic in \overline{X} . It follows that the direct limit is also arithmetic in \overline{X} , and uniformly so. \square

4.7. Canonical copies are not random for continuous measures. We now want to use the framework of ω -copies to show that for any $\alpha < \beta_N$, the canonical copy of a standard J -structure $\langle J_{\rho_\alpha^k}, A_\alpha^k \rangle$ cannot be K -random for a continuous measure, with K sufficiently large.

The argument rests mostly on various applications of the stair trainer technique introduced in Propositions 2.16 and 2.17, adapted to the notions of codings of countable J -structures developed in the previous sections. For convenience, we briefly review the core concepts.

ω -copy: A coding of a countable set-theoretic structure $\langle S, A \rangle$, $A \subseteq S$, as a subset of ω ; see Definitions 4.13 and 4.14.

Canonical copy: The copy of a J -structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ with projectum $\rho_\alpha^n = 1$ by means of a canonical bijection $V_\omega \leftrightarrow \omega$; see Definition 4.15.

Effective copy: An ω -copy of a J -structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ from which the internal fine structure hierarchy can effectively be recovered; see Definition 4.20 and Corollary 4.26. A canonical copy is always effective.

Pseudocopy: An ω -copy of a rud-closed structure that is an ω -model satisfying $\varphi_{V=L}$ and other J -like properties. The set of pseudocopies is arithmetically definable. A pseudocopy may be ill-founded. If it is well-founded, it codes some countable level J_α of the J -hierarchy; see Definition 4.37 and Lemma 4.39. By comparing their internal J -hierarchies, a subset of pseudocopies can be linearly ordered (up to isomorphism). This linear ordering is arithmetic, too; see Lemma 4.49.

We also fix some notation for the rest of this section. Given $N \in \omega$, we fix $c \in \omega$ to be a sufficiently large number. It will be greater than the complexity of all arithmetic definitions (N -pseudocopies, comparison of pseudocopies and S -operators, linearization) introduced in the previous sections. In particular,

Corollary 4.42 yields that, if X and Y are well-founded ω -copies of a J_α where $\alpha < \beta_N$, then the isomorphism between the two coded structures is recursive in $(X \oplus Y)^{(c)}$. It will also be greater than the respective constants from Corollary 4.43 and Lemma 4.50. After proving a couple of auxiliary results (Lemmas 4.51 and 4.52), we will also assume c to be greater than the constants appearing in these lemmas.

We give a first application of the stair trainer technique (as used in the proofs of Propositions 2.16 and 2.17) in the context of ω -copies and pseudocopies.

Lemma 4.51. *For each N , there exist numbers $d, e \in \omega$ such that the following holds. Suppose μ is a continuous measure, $\alpha < \beta_N$, and X is an ω -copy of J_α , recursive in $\mu^{(m)}$, for some $m \in \omega$. Suppose further that R is $(m + e)$ -random with respect to μ and computes an ω -copy of $J_{\alpha+1}$. Then there exists an ω -copy of $J_{\alpha+1}$ recursive in $\mu^{(m+d)}$.*

Proof. By Lemma 4.29, R computes an ω -copy of $S^{(n)}(J_\alpha)$, for all $n \in \omega$. By Corollary 4.43, we can fix a constant d_N so that $\bar{S}^{(n)}(X)$ is recursive in $(R \oplus \mu)^{(m+d_N)}$. By Lemma 4.30, we can fix a so that for any ω -copy of a transitive set, the a -th jump of the ω -copy computes its \bar{S} . Therefore $\bar{S}(X)$ is recursive in $\mu^{(m+a)}$. We may assume that R is $(m + d_N + a + 1)$ -random for μ . By Proposition 2.12 and Lemma 2.14, $\bar{S}(X)$ is recursive in $\mu^{(m+d_N)}$. Again by Lemma 4.30, $\bar{S}^{(2)}(X)$ is recursive in $\mu^{(m+d_N+a)}$. Since $\bar{S}^{(2)}(X)$ is also recursive in $(R \oplus \mu)^{(m+d_N)}$, we can apply Proposition 2.12 and Lemma 2.14 again and obtain that $\bar{S}^{(2)}(X)$ is recursive in $\mu^{(m+d_N)}$. We can continue inductively and obtain that for each $n \in \omega$, $\bar{S}^{(n)}(X)$ is recursive in $\mu^{(m+d_N)}$. Now apply Corollary 4.33. \square

The lemma shows that with the help of a sufficiently random real that computes an ω -copy of the next level of the J -hierarchy, μ can reach a copy of this level arithmetically, too. Combined with Lemma 4.50, this will be the key ingredient in proving that canonical copies of standard codes cannot be random with respect to a continuous measure.

The next lemma establishes a similar fact for the standard J -structures $\langle J_{\rho_\delta^n}, A_\delta^n \rangle$ over a given J_δ .

Lemma 4.52. *There exist numbers $d, e \in \omega$ such that the following holds. Suppose μ is a continuous measure and X is an ω -copy of J_δ . Further suppose X is recursive in $\mu^{(m)}$. Finally, suppose that R is $(m + e)$ -random*

with respect to μ and n is such that R computes an ω -copy of $\langle J_{\rho_\delta^n}, A_\delta^n \rangle$. Then there exists an ω -copy $\langle X_\delta^n, M_\delta^n \rangle$ of $\langle J_{\rho_\delta^n}, A_\delta^n \rangle$ recursive in $\mu^{(m+d)}$.

Proof. The proof is similar to that of Lemma 4.51, inductively using Proposition 2.12, Lemma 2.14, Corollary 4.35, and Corollary 4.42. We choose d large enough so that the isomorphism between X 's and R 's copy of J_δ is computable in $(\mu \oplus R)^{(m+d)}$. The number e then guarantees sufficient randomness in R to apply the stair trainer technique. \square

From now on, we assume that c is also greater than the respective constants from Lemmas 4.50, 4.51 and 4.52. We define $G(N) = (N + 2)(3c + 6)$.

Theorem 4.53. *Suppose $N \geq 0$, $\alpha < \beta_N$, and for some $m > 0$, $\rho_\alpha^m = 1$. Then the canonical copy of the standard J -structure $\langle J_{\rho_\alpha^m}, A_\alpha^m \rangle$ is not $G(N)$ -random with respect to any continuous measure.*

Proof. We fix R and m_R so that, when R is interpreted as a pair of reals, R is the canonical copy of some standard J -structure $\langle J_{\rho_\alpha^{m_R}}, A_\alpha^{m_R} \rangle$, where $\rho_\alpha^{m_R} = 1$. We assume for the sake of a contradiction that R is $G(N)$ -random with respect to a continuous measure μ .

To obtain a contradiction similar to the proofs of Propositions 2.16 and 2.17, we inductively define a hierarchy of indices of (pseudo)copies arithmetic in μ .

Definition 4.54. For each k with $0 \leq k \leq N$, we let

$$S_k = \{e \in \omega : \Phi_e^{\mu^{(k(3c+6))}} \text{ is total and } \Phi_e^{\mu^{(k(3c+6))}} \in \mathcal{PC}_N(\mu^{(k(3c+6))}) \\ \text{and for all } d < e, \Phi_d^{\mu^{(k(3c+6))}} \neq \Phi_e^{\mu^{(k(3c+6))}} \}.$$

The relation \preceq_N induces an ordering on the indices in each S_k , which will be denoted by \preceq_N , too. The linearly ordered subsets corresponding to \mathcal{PC}_N^* are given as

$$L_k = \{e \in S_k : \Phi_e^{\mu^{(k(3c+6))}} \in \mathcal{PC}_N^*(\mu^{(k(3c+6))})\}.$$

Finally, we let

$$I_k = \{e \in S_k : \Phi_e^{\mu^{(k(3c+6))}} \text{ is well-founded}\}.$$

By Lemma 4.49, the sets S_k and L_k are arithmetic in $\mu^{(k(3c+6))}$, for all $k \leq N$. In particular, by choice of c , L_k is recursive in $\mu^{(k(3c+6)+c)}$.

The following lemma shows that I_k is the longest well-founded initial segment of L_k .

Lemma 4.55. *Given $k \leq N$, let I be a well-founded initial segment of L_k . Then, for every $e \in I$, $\Phi_e^{\mu^{(k(3c+6))}}$ is a well-founded pseudocopy.*

Proof of Lemma 4.55. For any $i \in S_k$, denote by M_e the structure coded by $\Phi_e^{\mu^{(k(3c+6))}}$. Since I is well-founded, we can use induction on the ordinal height of I . Let $e \in I$. As induction hypothesis assume that for any $d \prec_N e$, M_d is well-founded. Since $\Phi_e^{\mu^{(k(3c+6))}}$ is a pseudocopy, M_e satisfies $\varphi_{V=L}$. Suppose x is an element of M_e . Since M_e satisfies $\varphi_{V=L}$, x must be in the rud-closure of a member y of M_e 's internal J -hierarchy. y is transitive in M_e . Similarly, the rud-closure of y is transitive in M_e (it may be all of M_e).

By Lemma 4.49, there is a $d \prec_N e$ such that y with the membership relation of M_e is isomorphic to M_d . The rud-closure of M_d is isomorphic to the rud-closure of y . Hence the rud-closure of y is well-founded. Therefore, x belongs to the rud-closure of y , which is transitive in M_e and well-founded. \square

Note that, conversely, if an ω -copy of J_β is recursive in $\mu^{(k(3c+6))}$, J_β will be indexed in L_k , since we only exclude ill-founded structures when passing to $\mathcal{PC}_N^*(\mu^{(k(3c+6))})$. Since it is well-founded, it will be indexed in I_k , too.

We will need an additional property of I_k .

Lemma 4.56. *If J_β is represented in I_k , then $\beta < \alpha$.*

Proof of Lemma 4.56. Suppose J_α is represented in I_k . By Lemma 4.52, there exists an ω -copy $\langle X, M \rangle$ of $\langle J_{\rho_\alpha^{m_R}}, A_\alpha^{m_R} \rangle$ recursive in $\mu^{(k(3c+6)+c)}$. (Recall m_R is such that $\rho_\alpha^{m_R} = 1$ and R represents $\langle J_1, A_\alpha^{m_R} \rangle$.) Comparing the canonical encoding of J_1 with X , we obtain that R is recursive in $\mu^{(k(3c+6)+2c)}$, contradicting the randomness of R . \square

We continue the proof of Theorem 4.53. We apply Lemma 2.15 to L_k to conclude I_k is recursive in $\mu^{(k(3c+6)+c+4)}$ as follows. Since R is a canonical copy, by Corollary 4.26 it computes an ω -copy of J_α . We can use that copy to test for any pseudocopy M with an index in L_k whether M embeds into J_α (using Corollary 4.46). This can be done recursively in $(\mu^{(k(3c+6))} \oplus R)^{(c)}$ by Lemma 4.49 and the choice of c . By Lemma 4.55, every pseudocopy with an index in I_k will embed into J_α . Consequently, I_k is recursive in $(\mu \oplus R)^{(k(3c+6)+c)}$. By choice of G , R is at least $(k(3c+6) + c + 5)$ -random for μ , so Lemma 2.15 implies that I_k is recursive in $\mu^{(k(3c+6)+c+4)}$.

Next we define by recursion a sequence of ordinals $\gamma_0, \dots, \gamma_K$, where K is at most $N + 1$.

- Let $\gamma_0 = \omega$ and $\xi_0 = 1$.

- Given γ_k , we check whether γ_k is a cardinal in each of the structures represented in I_k to which it belongs.
 - If so, we let

$$\gamma_{k+1} = \sup\{\beta: \exists e \in I_k (\beta \text{ has cardinality at most } \gamma_k \\ \text{in the structure indexed by } e)\},$$

$$\xi_{k+1} = \sup\{\beta: J_\beta \text{ is represented in } I_k\},$$

and we continue the recursion.

- Otherwise, γ_k is not a cardinal inside some structure J_δ represented in I_k . Since the recursion made it to step k , δ is greater than any β such that J_β has a representation in I_{k-1} . We terminate the recursion and let $K = k$.
- If we reach γ_{N+1} , we terminate the recursion.

By way of an example, consider γ_1 . It is the supremum of the ordinals which appear and are seen to be countable in one of the structures indexed in I_0 . Either it is the ω_1 of cofinally many of these structures or it is the supremum of their ordinals. If at step $k + 1$ there were a J_δ indexed in I_k in which γ_1 is countable, and (by a condensation argument for the J -hierarchy as in Section 4.2) every γ_j with $j \leq k$ is similarly countable, then the recursion ends. For β between γ_1 and the least such δ , $\gamma_1 = (\omega_1)^{J_\beta}$. More generally, for all $k \leq K$, and β between γ_k and the least δ as in the specification of the recursion, $\gamma_k = (\omega_k)^{J_\beta}$.

Lemma 4.57. *For every $0 \leq k \leq K$, $J_{\xi_{k+1}}$ is represented in I_k .*

Proof of Lemma 4.57. We first prove that J_2 is represented in I_0 . The canonical copy of J_1 is recursive. By Lemma 4.51, there exists an ω -copy of J_2 recursive in $\mu^{(c)}$. Note that $\rho_2 = 1$, hence the standard J -structure of J_2 is of the form $\langle J_1, A_2 \rangle$. The canonical copy of $\langle J_1, A_2 \rangle$ is recursive in R by Lemma 4.26. By Lemma 4.52, there exists an ω -copy of $\langle J_1, A_2 \rangle$ recursive in $\mu^{(2c)}$. The isomorphism between the J_1 of this copy and the canonical copy of J_1 is recursive in $\mu^{(3c)}$. Therefore, by Lemma 2.13, the canonical copy of $\langle J_1, A_2 \rangle$ is recursive in μ . By Corollary 4.26, there exists an ω -copy of J_2 recursive in μ .

Now assume $k > 0$. We show that J_{ξ_k} has a representation recursive in $\mu^{((k-1)(3c+6)+c)}$ as follows. If ξ_k is the maximum of the β for which J_β is represented in I_{k-1} , there is an ω -copy of J_{ξ_k} recursive in $\mu^{((k-1)(3c+6))}$.

Otherwise, since I_{k-1} is recursive in $\mu^{(k-1)(3c+6)+4}$, Lemma 4.50 implies that there exists an ω -copy of J_{ξ_k} recursive in $\mu^{((k-1)(3c+6)+4+c)}$.

Next, we can apply Lemma 4.51 to obtain an ω -copy of $J_{\xi_{k+1}}$ recursive in $\mu^{((k-1)(3c+6)+4+2c)}$, which implies that it recursive in $\mu^{k(3c+6)}$ and hence that it is represented in I_k . \square

The lemma implies that for each $i < K$, γ_i appears as a cardinal in some structure represented in I_i .

Case 1: $K < N + 1$, that is, the recursion terminates early. Let δ be fixed to be least as it appeared in the definition of the γ and ξ sequences.

In this case either the ultimate projectum ρ_δ of J_δ is 1 or there is an $1 < i < K$ such that ρ_δ is γ_i . This is because for every infinite ordinal less than γ_K , there is a structure represented in I_{K-1} in which this ordinal is in one-to-one correspondence with some γ_i , $i < K$.

Case 1a: $\rho_\delta = 1$. The canonical copy of J_1 is recursive. By Lemma 4.52, there exists an ω -copy of $\langle J_1, A_\delta \rangle$ recursive in $\mu^{(K(3c+6)+c)}$. Lemma 4.18 implies that the canonical copy $\langle X, M \rangle$ of $\langle J_1, A_\delta \rangle$ is recursive in $\mu^{(K(3c+6)+c+1)}$. Since $\alpha > \delta$, R computes $\langle X, M \rangle$, by Lemma 4.28. By Lemma 2.13, μ computes $\langle X, M \rangle$. Since $\langle X, M \rangle$ is an effective copy, Corollary 4.26 implies μ computes an ω -copy of J_δ . But this means J_δ is represented in I_0 , which contradicts the definition of γ_1 .

Case 1b: $\rho_\delta = \gamma_i > 1$. We established above that I_{i-1} is recursive in $\mu^{((i-1)(3c+6)+c+4)}$. Then by Lemma 4.50, there exists an ω -copy Y of J_{ρ_δ} recursive in $\mu^{((i-1)(3c+6)+2c+4)}$.

Since J_δ is represented in I_K , there exists an ω -copy of J_δ recursive in $\mu^{(K(3c+6))}$. By Lemma 4.52, there exists an ω -copy $\langle X, M_X \rangle$ of $\langle J_{\rho_\delta}, A_\delta \rangle$ recursive in $\mu^{(K(3c+6)+c)}$. We think of Y as a simple representation of J_{ρ_δ} relative to μ , whereas X is a complicated representation.

By Lemma 4.45 and choice of c , we can retrieve the isomorphism between X 's and Y 's representations of J_{ρ_δ} recursively in $\mu^{(K(3c+6)+c+c)} = \mu^{(K(3c+6)+2c)}$ and transfer M_X (the coding of A_δ in X) to Y . This gives us an ω -copy $\langle Y, M_Y \rangle$ of $\langle J_{\rho_\delta}, A_\delta \rangle$ recursive in $\mu^{(K(3c+6)+2c)}$.

Since $\alpha > \delta$, R computes, by Lemma 4.28, another ω -copy of $\langle J_{\rho_\delta}, A_\delta \rangle$, say $\langle Z, M_Z \rangle$. This copy is trivial relative to R . Using at most c jumps, the join of R and $\mu^{((i-1)(3c+6)+2c+4)}$ can compare Y and Z and map M_Z , the encoding of A_δ in Z , to M_Y , the encoding of A_δ in Y . This way we obtain that the ω -copy $\langle Y, M_Y \rangle$ of $\langle J_{\rho_\delta}, A_\delta \rangle$ recursive in $(R \oplus \mu^{((i-1)(3c+6)+2c+4)})^{(c)}$,

i.e. recursive in $R \oplus \mu^{((i-1)(3c+6)+3c+4)}$. By Lemma 2.14, $\langle Y, M_Y \rangle$ is recursive in $\mu^{((i-1)(3c+6)+3c+4)}$. The complicated predicate M_Y on the simple structure Y is in fact relatively simple.

By Corollary 4.27, $\mu^{((i-1)(3c+6)+3c+4+2)} = \mu^{(i(3c+6))}$ computes an ω -copy of J_δ . But this implies J_δ is represented in S_i . In particular, J_δ is represented in I_i . This contradicts the fact that δ is greater than any β such that J_β has a representation in I_{K-1} .

Case 2: $K = N + 1$.

The analysis is similar to Case 1. γ_{N+1} is defined. Since R cannot be represented in any S_N , $\gamma_{N+1} < \beta_N$. Hence γ_{N+1} is not a cardinal in $J_{\gamma_{N+1}}$ and thus $J_{\gamma_{N+1}}$ must be projectable, say to γ_i . There is an ω -copy of $J_{\gamma_{N+1}}$ recursive in $\mu^{(N(3c+6)+2c+4)}$. By the same argument as above, there is a ω -copy of $\langle J_{\gamma_i}, A_{\gamma_{N+1}} \rangle$ recursive in $\mu^{((i-1)(3c+6)+3c+4)}$, which in turn yields a ω -copy of $J_{\gamma_{N+1}}$ recursive in $\mu^{(i(3c+6))}$. This contradicts that $J_{\gamma_{N+1}}$ is not represented in any I_i , for $i \leq N$.

This is sufficient to complete the proof of Theorem 4.53. □

4.8. Finishing the proof of Theorem 2. We restate Theorem 2. Let $G(n)$ be the recursive function defined before the statement of Theorem 4.53 in Section 4.7.

Theorem 2. *For every $n \in \omega$,*

$$\text{ZFC}_n^- \not\vdash \text{“NCR}_{G(n)} \text{ is countable.”}$$

Proof. By Theorem 4.53, the set \mathcal{X} of canonical copies of standard J -structures is a subset of $\text{NCR}_{G(n)}$. The non-existence of the representation of a measure with an arithmetic property is downwards absolute between ω -models. Therefore, to prove the theorem, it is sufficient to show that \mathcal{X} is not countable in L_{β_n} .

Every real inside L_{β_n} is computable from some element in \mathcal{X} . This follows from Jensen’s analysis of projecta: when a new real is constructed so is a subset of ω that is a standard code for the J -structure in which the real appears. By the usual Cantor diagonal argument, the set of reals in L_{β_n} is uncountable in L_{β_n} , so \mathcal{X} is not countable in L_{β_n} , too. □

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