Lecture Notes on Descriptive Set Theory

Jan Reimann*

Department of Mathematics
Pennsylvania State University

Notation

$U_\varepsilon(x)$	Ball of radius ε about x
$\overline{U}$	Topological closure of U
$2^{<\mathbb{N}}$	Set of finite binary strings
$\sigma, \tau, \dots$	Finite binary strings
$2^{\mathbb{N}}$	Cantor space, set of all infinite binary sequences
$\mathbb{N}^{\mathbb{N}}$	Baire space, set of all infinite sequences of natural numbers
$\alpha\lvert_n$	Length n initial segment of sequence α, $\alpha(0)\dots\alpha(n-1)$
N_σ	Open cylinder defined by σ
$\mathrm{Lip}_L(X)$	Set of L-Lipschitz functions on X
$\mathrm{diam}(X)$	Diameter of a set X in a metric space, $\mathrm{diam}(X) = \sup\{d(x,y) \colon x, y \in X\}$

*reimann@math.psu.edu

Lecture 1: Perfect Subsets of the Real Line

Descriptive set theory nowadays is understood as the study of definable subsets of Polish Spaces. Many of its problems and techniques arose out of efforts to answer basic questions about the real numbers. A prominent example is the *Continuum Hypothesis* (CH):

> If $A \subseteq \mathbb{R}$ is uncountable, does there exist a bijection between A and $\mathbb{R}$? That is, is every uncountable subset of $\mathbb{R}$ of the same cardinality as $\mathbb{R}$? [Cantor, 1890's]

Early approaches to this problem tried to show that CH holds for a number of sets with an easy topological structure. It is a standard exercise of analysis to show that every open set satisfies CH. (An open set contains an interval, which maps bijectively to $\mathbb{R}$.) For closed sets, the situation is less clear. Given a set $A \subseteq \mathbb{R}$, we call $x \in \mathbb{R}$ an *accumulation point* of A if

$$\forall \epsilon > 0 \, \exists z \in A \, [z \neq x \;\&\; z \in U_\varepsilon(x)],$$

where $U_\varepsilon(x)$ denotes the standard ε-neighborhood of x in $\mathbb{R}$[1]. Call a non-empty set $P \subseteq \mathbb{R}$ *perfect* if it is closed and every point of P is an accumulation point. In other words, a perfect set is a closed set that has no *isolated* points. It is not hard to see that for a perfect set P, every neighborhood of a point $p \in P$ contains infinitely many points.

Obviously, $\mathbb{R}$ itself is perfect, as is any closed interval in $\mathbb{R}$. There are totally disconnected perfect sets, such as the *middle-third Cantor set* in $[0,1]$

Theorem 1.1: *A perfect subset of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$.*

Proof. Let $P \subseteq \mathbb{R}$ be perfect. We construct an injection from the set $2^{\mathbb{N}}$ of all infinite binary sequences into P. An infinite binary sequence $\xi = \xi_0\xi_1\xi_2\ldots$ can be identified with a real number $\in [0,1]$ via the mapping

$$\xi \mapsto \sum_{i \geq 0} \xi_i 2^{-i-1}.$$

Note that this mapping is onto. Hence the cardinality of P is at least as large as the cardinality of $[0,1]$. The **Cantor-Schröder-Bernstein Theorem** (for a proof see e.g. **(author?)** [Jec03]) implies that $|P| = 2^{\aleph_0}$.

[1] There are some divergences in terminology. Some authors call an accumulation point a limit point. We reserve the latter term for any point that is the limit of a sequence of points from a given set. Hence every member of a set is a limit point of that set. In particular, isolated members of a set are limit points.

Choose $x \in P$, and let $\varepsilon_0 = 1 = 2^0$. Since P is perfect, $P \cap U_{\varepsilon_0}(x)$. Let $x_0 \neq x_1$ be two points in $P \cap U_{\varepsilon_0}(x)$, distinct from x. Let ε_1 be such that $\varepsilon_1 \leq 1/2$, $U_{\varepsilon_1}(x_0), U_{\varepsilon_1}(x_1) \subseteq U_{\varepsilon_0}(x)$, and $\overline{U_{\varepsilon_1}(x_0)} \cap \overline{U_{\varepsilon_1}(x_1)} = \emptyset$, where $\overline{U}$ denotes the closure of U.

We can iterate this procedure recursively with smaller and smaller diameters, using the fact that P is perfect. This gives rise to a so-called *Cantor scheme*, a family of open balls (U_σ). Here the index σ is a finite binary sequence, also called a *string*. The scheme has the following properties.

C1) $\mathrm{diam}(U_\sigma) \leq 2^{-|\sigma|}$, where $|\sigma|$ denotes the length of σ.

C2) If τ is a proper extension of σ, then $\overline{U_\tau} \subset U_\sigma$.

C3) If τ and σ are *incompatible* (i.e. neither extends the other), then

$$\overline{U_\tau} \cap \overline{U_\sigma} = \emptyset.$$

C4) The center of each U_σ, call it x_σ, is in P.

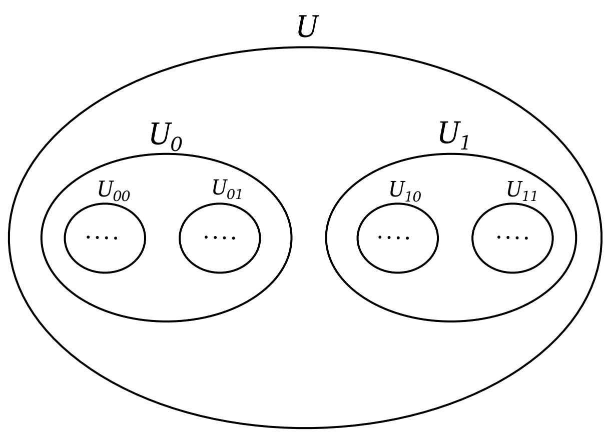

Figure 1: Cantor Scheme

Let ξ be an infinite binary sequence. Given $n \geq 0$, we denote by $\xi\,|_n$ the string formed by the first n bits of ξ, i.e.

$$\xi\,|_n = \xi_0 \xi_1 \ldots \xi_{n-1}.$$

The finite initial segments give rise to a sequence $x_{\xi|_n}$ of centers. By (C1) and (C2), this is a Cauchy sequence. By (C4), the sequence lies in P. Since P is closed, the limit x_ξ is in P. By (C3), the mapping $\xi \mapsto x_\xi$ is well-defined and injective. $\square$

Theorem 1.2: *Every uncountable closed subset of $\mathbb{R}$ contains a perfect subset.*

Proof. Let $C \subseteq \mathbb{R}$ be uncountable and closed. We say $z \in \mathbb{R}$ is a *condensation point* of C if

$$\forall \varepsilon > 0\, [U_\varepsilon(z) \cap C \text{ uncountable}].$$

Let D be the set of all condensation points of C. Note that $D \subseteq C$, since every condensation point is clearly an accumulation point and C is closed. Furthermore, we claim that D is perfect. Clearly D is closed. Suppose $z \in D$ and $\varepsilon > 0$. Then $U_\varepsilon(z) \cap C$ is uncountable. We would like to conclude that $U_\varepsilon(z) \cap D$ is uncountable, too, since this would mean in particular that $U_\varepsilon(z) \cap D$ is infinite. The conclusion holds if $C \setminus D$ is countable. To show that $C \setminus D$ is countable, we use the fact that every open interval in $\mathbb{R}$ is the union of countably many open intervals with rational endpoints. Note that there are only countably many such intervals. If $y \in C \setminus D$, then for some $\delta > 0$, $U_\delta(y) \cap C$ is countable. y is contained in some subinterval $U_y \subseteq U_\delta(y)$ with rational endpoints. Thus, we have

$$C \setminus D \subseteq \bigcup_{y \in C \setminus D} U_y \cap C,$$

and the right hand side is a countable union of countable sets, hence countable. $\square$

We will later encounter an alternative (more constructive) proof that gives additional information about the complexity of the closed set C. For now we conclude with the fact we started out to prove.

Corollary 1.3: *Every closed subset of $\mathbb{R}$ is either countable or of the cardinality of the continuum.*

Lecture 2: Polish Spaces

The proofs in the previous lecture are quite general, that is, they make little use of specific properties of $\mathbb{R}$. If we scan the arguments carefully, we see that we can replace $\mathbb{R}$ by any metric space that is *complete and contains a countable basis of the topology*.

Review of some concepts from topology

Let $(X, \mathcal{O})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{O}$ of subsets if X is a **basis** for the topology if every open set from $\mathcal{O}$ is the union of elements of $\mathcal{B}$. For example, the open intervals with rational endpoints form a basis of the standard topology of $\mathbb{R}$. (We used this fact in Lecture 1.) $\mathcal{S} \subseteq \mathcal{O}$ is a **subbasis** if the set of finite intersections of sets in $\mathcal{S}$ is a basis for the topology. Finally, if $\mathcal{S}$ is any family of subset of X, the **topology generated by** $\mathcal{S}$ is the smallest topology on X containing $\mathcal{S}$. It consists of all unions of finite intersections of sets in $\mathcal{S} \cup \{X, \emptyset\}$.

A set $D \subset X$ is **dense** if for open $U \neq \emptyset$ there exists $z \in D \cap U$. If a topological space $(X, \mathcal{O})$ has a countable dense subset, the space is called **separable**.

If $(X_i)_{i \in I}$ is a family of topological spaces, one defines the **product topology** on $\Pi_{i \in I} X_i$ to be the topology generated by the sets $\pi_i^{-1}(U)$, where $i \in I$, $U \subseteq X_i$ is open, and $\pi_i : \Pi_{i \in I} X_i \to X_i$ is the ith projection.

Now suppose (X, d) is a metric space. With each point $x \in X$ and every $\varepsilon > 0$ we associate an **ε-neighborhood** or **ε-ball**

$$U_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}.$$

The ε-neighborhoods form the basis of a topology, called the **topology** of the metric space (X, d). If this topology agrees with a given topology $\mathcal{O}$ on X, we say the metric d is **compatible** with the topology $\mathcal{O}$. If for a topological space $(X, \mathcal{O})$ there exists a compatible metric, $(X, \mathcal{O})$ is called **metrizable**[2].

If a topological space $(X, \mathcal{O})$ is separable and metrizable, then the balls with center in a countable dense subset D and rational radius form a *countable base of the topology*.

[2]Note that a compatible metric is not necessarily unique.

Polish spaces

Definition 2.1: A **Polish space** is a separable topological space X for which exists a compatible metric d such that (X, d) is a complete metric space.

As mentioned before, there may be many different compatible metrics that make X complete. If X is already given as a complete metric space with countable dense subset, then we call X a **Polish metric space**.

The standard example is, of course, $\mathbb{R}$, the set of real numbers. One can obtain other Polish spaces using the following basic observations.

Proposition 2.2:

1) *A closed subset of a Polish space is Polish.*
2) *The product of a countable (in particular, finite) sequence of Polish spaces is Polish.*

Hence we can conclude that $\mathbb{R}^n$, $\mathbb{C}$, $\mathbb{C}^n$, the unit interval $[0,1]$, the unit circle $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, and the infinite dimensional spaces $\mathbb{R}^{\mathbb{N}}$ and $[0,1]^{\mathbb{N}}$ (the *Hilbert cube*) are Polish spaces.

Any countable set with the **discrete topology** is Polish, by means of the **discrete metric** $d(x, y) = 1 \iff x \neq y$.

Some subsets of Polish spaces are Polish but not closed. For example, $(0,1)$, the open unit interval, is a Polish space, of course with a different metric. We will later characterize all subsets of Polish spaces that are Polish themselves.

Product spaces

In a certain sense, the most important Polish spaces are of the form $A^{\mathbb{N}}$, where A is a countable set carrying the discrete topology. The standard cases are

$$2^{\mathbb{N}}, \text{ the Cantor space} \qquad \text{and} \qquad \mathbb{N}^{\mathbb{N}}, \text{ the Baire space.}$$

We will, for now, denote elements from $A^{\mathbb{N}}$ by lower case greek letters from the beginning of the alphabet. The n-th term of α we will denote by either $\alpha(n)$ or α_n, whichever is more convenient.

We endow A with the discrete topology. The product topology on these spaces has a convenient characterization. Given a set A, let $A^{<\mathbb{N}}$ be the sets of all finite binary sequences over A. Given $\sigma, \tau \in A^{\mathbb{N}}$, we write $\sigma \subset \tau$ to indicate that σ is an initial segment of τ. $\subset$ means the initial segment is proper. This notation extends naturally to hold between elements of $2^{<\mathbb{N}}A$ and $A^{\mathbb{N}}$, $\sigma \subset \alpha$ meaning that σ is a finite initial segment of α.

A basis for the product topology on $A^{\mathbb{N}}$ is given by the **cylinder sets**

$$N_\sigma = \{\alpha \in A^{\mathbb{N}} : \sigma \subset \alpha\},$$

that is, the set of all infinite sequences extending σ. The complement of a cylinder is a finite union of cylinders and hence open. Therefore, each set N_σ is clopen.

A compatible metric is given by

$$d(\alpha, \beta) = \begin{cases} 2^{-N} & \text{where } N \text{ is least such that } \alpha_N \neq \beta_N \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

The representation of the topology via cylinders (which are characterized by finitary objects) allows for a combinatorial treatment of many questions and will be essential later on.

Proposition 2.3 (Topological properties of $A^{\mathbb{N}}$)**:** *Let A be a countable set, equipped with the discrete topology. Suppose $A^{\mathbb{N}}$ is equipped with the product topology. Then the following hold.*

1) *$A^{\mathbb{N}}$ is Polish.*
2) *$A^{\mathbb{N}}$ is* zero-dimensional*, i.e. it has a basis of* clopen *sets.*
3) *$A^{\mathbb{N}}$ is compact if and only if A is finite.*

Via the mapping

$$\alpha \mapsto \sum_{i=0}^{\infty} \frac{2\alpha_i}{3^{i+1}},$$

$2^{\mathbb{N}}$ is homeomorphic to the middle-third Cantor set in $\mathbb{R}$, whereas the **continued fraction** mapping

$$\beta \mapsto \beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{\beta_2 + \cfrac{1}{\beta_3 + \dots}}}$$

provides a homeomorphism between $\mathbb{N}^{\mathbb{N}}$ and the irrational real numbers.

The universal role played by the discrete product spaces is manifested in the following results.

Theorem 2.4: *Every uncountable Polish space contains a homeomorphic embedding of the Cantor space,* $2^{\mathbb{N}}$.

The proof is similar to the proof of Theorem 1.1. Note that the proof actually constructs an embedding of $2^{\mathbb{N}}$. The continuity of the mapping is straightforward.

In a similar way we can adapt the proof of Theorem 1.2 to show that the *prefect subset property* holds for closed subsets of Polish spaces.

Theorem 2.5 (Cantor-Bendixson Theorem for Polish spaces)**:** *Every uncountable closed subset of a Polish space contains a perfect subset.*

Finally, we can characterize Polish spaces as continuous images of Baire space.

Theorem 2.6: *Every Polish space X is the continuous image of the Baire space,* $\mathbb{N}^{\mathbb{N}}$.

Proof. Let d be a compatible metric on X, and let $D = \{x_i : i \in \mathbb{N}\}$ be a countable dense subset of X. Every point in X is the limit of a sequence in D. Define a mapping $g : \mathbb{N}^{\mathbb{N}} \to X$ by putting

$$\alpha = \alpha(0)\,\alpha(1)\,\alpha(2) \cdots \mapsto \lim_n x_{\alpha(n)}.$$

The problem is, of course, that the limit on the right hand side not necessarily exists. Besides, even if it exists, the mapping may not be continuous at that point, since we made no additional assumptions about the set D.

To remedy the situation, we proceed more carefully. Given $\alpha \in \mathbb{N}$, we define iteratively $y_0^\alpha = x_{\alpha(0)}$ and

$$y_{n+1}^\alpha = \begin{cases} x_{\alpha(n+1)} & \text{if } d(y_n^\alpha, x_{\alpha(n+1)}) < 2^{-n}, \\ y_n^\alpha & \text{otherwise .} \end{cases}$$

The resulting sequence (y_n^α) is clearly Cauchy in X, and hence converges to some point $y^\alpha \in X$, by completeness. We define

$$f(\alpha) = y^\alpha.$$

f is continuous, since if α and β agree up to length N (that is, their distance is at most 2^{-N} with respect to the above metric), then the sequences (y_n^α) and

(y_n^β) will agree up to index N, and all further terms are within 2^{-N} of y_N^α and y_N^β, respectively.

Finally, since D is dense in X, f is a surjection. $\square$

Lecture 3: Excursion – The Urysohn Space

Recall that a mapping $e : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is an **isometry** if

$$d_Y(f(x), f(y)) = d_X(x, y) \quad \text{for all } x, y \in X,$$

that is, an isometry is a mapping that preserves distances. f is also called an *isometric embedding* of X into Y. X and Y are **isometric** if there exists a bijective isometry between them.

It is a remarkable fact that there exists a "universal" Polish space – a complete, separable metric space that contains an *isometric copy* of any other Polish metric space.

Theorem 3.1: *There exists a Polish metric space $\mathbb{U}$ such that every Polish metric space isometrically embeds into $\mathbb{U}$.*

A concrete example of such a space is $\mathcal{C}[0,1]$, the set of all continuous real-valued functions on $[0,1]$ with the sup-metric (see exercises). But this space is not quite what we have in mind. There exists another space with a stronger, more "intrinsic" universality property, **Urysohn space**. It was constructed by **(author?)** [Ury27].

The construction features an *amalgamation principle* that has surfaced in other areas like model theory or graph theory. It has recently attracted increased attention, which has also led to renewed interest in the Urysohn space.

Extensions of finite isometries and Urysohn universality

We first sketch the basic idea for constructing the Urysohn space. Suppose X is a Polish metric space. Let $D = \{x_1, x_2, \dots\}$ be a countable, dense subset. We first observe that it is sufficient to isometrically embed D into $\mathbb{U}$.

Lemma 3.2: *If Y is Polish, then any isometric embedding e of D into Y extends to an isometric embedding e^* of X into Y.*

Proof. Given $z \in X$, let (x_{i_n}) be a sequence in D converging to z. Since (x_{i_n}) converges, it is Cauchy. e is an isometry, and thus $y_n := e(x_{i_n})$ is Cauchy, and since Y is Polish, (y_n) converges to some $y \in Y$. Put $e^*(z) = y$. To see that this mapping is well-defined, let (x_{j_n}) be another sequence with $x_{j_n} \to z$.

Then $d(x_{i_n}, x_{j_n}) \to 0$, and hence $d(e(x_{i_n}), e(x_{j_n}) = d(y_n, e(x_{j_n})) \to 0$, implying $e(x_{j_n}) \to y$. Furthermore, suppose $w = \lim x_{k_n}$ is another point in X. Then (since a metric is a continuous mapping from $Y \times Y \to \mathbb{R}$)

$$d(e*(z), e^*(w)) = \lim d(e(x_{i_n}), e(x_{k_n})) = \lim d(x_{i_n}, x_{k_n}) = d(z, w).$$

Thus e^* is an isometry. □

In order to embed D, we can now exploit the *inductive structure* of $\mathbb{N}$ and reduce the task to *extending finite isometries*.

Suppose we have constructed an isometry e between $F_N = \{x_1, \dots, x_N\} \subset D$ and $\mathbb{U}$. We would like to extend the isometry to include x_{N+1}. For this we have to find an element $y \in \mathbb{U}$ such that for all $i \leq N$

$$d_{\mathbb{U}}(y, e(x_i)) = d_X(x_{N+1}, x_i).$$

This extension property gives rise to the following definition.

Definition 3.3: A Polish space $(\mathbb{U}, d_{\mathbb{U}})$ is **Urysohn universal** if for every finite subspace $F \subset \mathbb{U}$ and any extension $F^* = F \sqcup \{x^*\}$ with metric d^* such that

$$d^*|_{X \times X} = d_{\mathbb{U}},$$

there exists a point $u \in \mathbb{U}$ such that

$$d_{\mathbb{U}}(u, x) = d^*(x^*, x) \quad \text{for all } x \in F.$$

One can show that any two Urysohn universal spaces are isometric. We will show here that this unique (up to isometry) space actually exists, the **Urysohn space** $\mathbb{U}$.

The extension property also implies a strong intrinsic extension property for the Urysohn space itself.

Proposition 3.4: *Let $\mathcal{U}$ be a separable and complete metric space that contains an isometric image of every separable metric space. Then $\mathcal{U}$ is Urysohn universal if and only if every isometry between finite subsets of $\mathcal{U}$ extends to an isometry of $\mathcal{U}$ onto itself.*

Constructing the Urysohn space – a first approximation

We first give a construction of a space that has the extension property, but is not Polish. After that we will take additional steps to turn it into a Polish space.

The crucial idea is to observe that if X is a metric space and $x \in X$, then the mapping $f_x : X \to \mathbb{R}^{\geq 0}$ given by

$$f_x(y) = d_X(x, y)$$

is 1-*Lipschitz*. Recall that a function g between metric spaces X and Y is **L-Lipschitz**, $L > 0$ if for every $x, y \in X$,

$$d(g(x), g(y)) \leq L\, d(x, y).$$

Let $\mathrm{Lip}_1(X)$ be the set of 1-Lipschitz mappings from X to $\mathbb{R}$. We endow $\mathrm{Lip}_1(X)$ with the *supremum* metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}.$$

If $\mathrm{diam}(X) \leq \mathrm{d}$ and f, g are 1-Lipschitz, then $d(f, g)$ is indeed finite. However, we will need that the resulting space is also bounded. Let $\mathrm{Lip}_1^{\mathrm{d}}(X)$ be the space of all 1-Lipschitz functions from X to $[0, \mathrm{d}]$. Clearly, $\mathrm{diam}(\mathrm{Lip}_1^{\mathrm{d}}(X)) \leq \mathrm{d}$.

With this metric, the mapping $x \mapsto f_x(y) = d(x, y)$ becomes an isometry: We have

$$d(f_x, f_z) = \sup\{|d(x, y) - d(z, y)| : y \in X\}.$$

By the reverse triangle inequality, this is always $\leq d(x, z)$. On the other hand, setting $z = x$ yields $d(f_x, f_z) \geq d(x, z)$. This embedding of X into $\mathrm{Lip}_1^{\mathrm{d}}(X)$ is called the **Kuratowski embedding**.

We use this fact as follows: If $X^* = X \sqcup \{x^*\}$ and d^* is an extension of d_X, then f_{x^*} is an element of $\mathrm{Lip}_1^{\mathrm{d}}(X)$, and as above, for any $x \in X$

$$d(f_{x^*}, f_x) = d^*(x^*, x).$$

Hence $\mathrm{Lip}_1^{\mathrm{d}}(X)$ has an extension property of the kind we are looking for.

Iterative construction: Let X_0 be any non-empty Polish space with finite diameter $\mathrm{d} > 0$. Given X_n, let $\mathrm{d}(n) = \mathrm{diam}(X_n)$ and set $X_{n+1} = \mathrm{Lip}_1^{2\mathrm{d}(n)}(X_n)$. Finally, put $X_\infty = \bigcup_n X_n$. Note that X_∞ inherits a well-defined metric d from the X_n, which embed isometrically into it.

We claim that X_∞ has the extension property needed to be Urysohn universal. Let F be a finite subset of X_∞. There exists N such that $F \subset X_N$. Suppose $F^* = F \sqcup \{x^*\}$ and d^* is an extension of d to F^*. Let $\mathrm{d}^* = \mathrm{diam}(F^*)$. Note that $\mathrm{diam}(X_n) = 2^n \mathrm{d}$. Choose M so that $M \geq N$ and $\mathrm{diam}(X_M) \geq \mathrm{d}^*$. The next lemma ensures that we can find $f \in X_{M+1}$ such that $f(x) = d^*(x^*, x)$ for all $x \in F$.

Lemma 3.5 (McShane-Whitney)**:** *Let X be a metric space with* $\mathrm{diam}(X) \leq \mathrm{d}$*, $A \subseteq X$, and $f \in \mathrm{Lip}_1^{\mathrm{d}}(A)$, then f can be extended to an 1-Lipschitz function f^* on all of X such that*

$$f^*|_A = f \quad \textit{and} \quad f^* \in \mathrm{Lip}_1^{2\mathrm{d}}(X).$$

Proof. For each $a \in A$ define $f_a : X \to \mathbb{R}$ as

$$f_a(x) = f(a) + d(a, x).$$

Then f_a is 1-Lipschitz, by the reverse triangle inequality. Let

$$f^*(x) = \inf\{f_a(x) \colon a \in A\}.$$

Then $f^*(a) = f(a)$ for all $a \in A$. Let $x, y \in X$ and $\varepsilon > 0$. Wlog assume $f^*(y) \geq f^*(x)$. Pick $a \in A$ so that $f_a(x) \leq f^*(x) + \varepsilon$. Then

$$\begin{aligned} |f^*(x) - f^*(y)| = f^*(y) - f^*(x) &\leq f^*(y) - f_a(x) + \varepsilon \\ &\leq f_a(y) - f_a(x) + \varepsilon \leq Ld(x, y) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $|f^*(x) - f^*(y)| \leq Ld(x, y)$.

Finally, we have $f(a) \leq f_a(x) \leq f(a) + \mathrm{d}$ and thus $0 \leq f^*(x) \leq f_a(x) \leq 2\mathrm{d}$. □

Mending the construction

The set X_∞ we constructed has two deficiencies with respect to our goal of constructing a Urysohn universal space: X_∞ is not necessarily separable, and X_∞ is not necessarily complete.

To make X_∞ separable, we observe that if X is compact, then the set $\mathrm{Lip}_1^{\mathrm{d}}(X)$ is closed in $\mathcal{C}(X)$ (the set of all real-valued continuous functions on X), bounded, and *equicontinuous*. By the *Arzelà-Ascoli Theorem*, $\mathrm{Lip}_1^{\mathrm{d}}(X)$ is compact. Every compact metric space is separable: For every $\varepsilon > 0$, there exists a finite covering of the space with sets of diam $< \varepsilon$. Letting ε traverse all positive rationals and picking a point from each set in an ε-covering yields a countable dense subset.

Hence if we start with X_0 compact, each X_n will be compact, too. A countable union of separable spaces is separable, thus X_∞ is separable.

To obtain a complete space, we can pass from X_∞ to its *completion* $\overline{X_\infty}$. First note that if a metric space X is separable, so is its completion $\overline{X}$. However, we also have to ensure that $\overline{X_\infty}$ retains the universality property of X_∞.

Lemma 3.6: *If a complete metric space (Y, d) admits a dense Urysohn universal subspace $\mathcal{U}$, then Y is Urysohn universal.*

Proof. We follow **(author?)** [Gro99]. Let $F = \{x_1, \dots, x_n\} \subset Y$ and assume $F^* = F \sqcup \{x^*\}$ is an extension with metric d^*.

We first note that Y is *aproximately universal*. This means that for any $\varepsilon > 0$, there exists a point $y^* \in Y$ such that

$$|d(y*, x) - d^*(x^*, x)| < \varepsilon \quad \text{for all } x \in F. \qquad (*)$$

This can be seen as follows. Since $\mathcal{U}$ is dense in Y, we can find a finite set $F_\varepsilon = \{z_1, \dots, z_n\} \subset \mathcal{U}$ such that

$$d(x_i, z_i) < \varepsilon \quad \text{for } 1 \le i \le n.$$

To keep the proof technically simple, wlog we assume ε is much smaller than the individual distances between the x_i. Consider the extension $F_\varepsilon^* = F_\varepsilon \sqcup \{x^*\}$ with metric

$$e^*(x^*, z_i) = d^*(x^*, x_i) + d(x_i, z_i).$$

Since $\mathcal{U}$ has the finite extension property, we can find $y^* \in \mathcal{U}$ such that

$$d(y^*, z_i) = e^*(x^*, z_i)$$

Hence

$$\begin{aligned} |d(y^*, x_i) - d^*(x^*, x_i)| &= |e^*(x^*, z_i) - d^*(x^*, x_i)| \\ &= |d^*(x^*, x_i) + d(x_i, z_i) - d^*(x^*, x_i)| < \varepsilon. \end{aligned}$$

We use this approximate universality to construct a Cauchy sequence (y_k) in Y of ‘approximate’ extension points that satisfy $(*)$ for smaller and smaller ε.

Let $0 < \delta = \max\{d^*(x^*, x_i) \colon 1 \le i \le n\}$. The formal requirements for the sequence (y_i) are as follows.

(i) $|d(y_k, x_i) - d^*(x^*, x_i)| \le 2^{-k}\delta$.

(ii) $d(x_{k+1}, x_k) \leq 2^{-k}\delta$.

The sequence necessarily converges in Y and the limit point must be a true extension point, due to (i).

Suppose we have already constructed $y_1, \ldots, y_k$ satisfying (i), (ii). Add an (abstract) point y^*_{k+1} to $F_k = F \cup \{y_1, \ldots, y_k\}$. Let $F^*_{k+1} = F_k \sqcup \{y^*_{k+1}\}$.

We want to use approximate universality on F^*_{k+1}. To this end we have to define a metric e^* on F^*_{k+1} that has the following properties

$$e^*|_{F_k} = d|_{F_k} \tag{+}$$

$$e^*(y^*_{k+1}, x_i) = d^*(x^*, x_i) \quad (1 \leq i \leq n) \tag{++}$$

$$e^*(y^*_{k+1}, y_k) = 2^{-k-1}\delta \tag{+++}$$

Indeed such a metric exists: The condition (+) already defines a metric on the set F_k. (+)-(+ + +) also define a metric on $F \cup \{y_k, y^*_{k+1}\}$. The only thing left to check for this is the triangle inequality for y_k, y^*_{k+1}.

$$|e^*(x_i, y_k) - e^*(y^*_{k+1}, x_i)| = |d(x_i, y_k) - d^*(x^*, x_i)| \leq 2^{-k}\delta = e^*(y_k, y^*_{k+1}),$$

by (i). These metrics agree on the set

$$F_k \cap (F \cup \{y_k, y^*_{k+1}\}) = F \cup \{y_k\}.$$

Therefore, we can "merge" them to a metric on all of F^*_{k+1} by letting

$$e^*(y^*_{k+1}, y_j) = \inf\{e^*(y^*_{k+1}, z) + e^*(z, y_j) : z \in \{y_1, \ldots, y_{k-1}\}\}.$$

Now choose $\varepsilon < 2^{-k-1}\delta$ and apply approximate universality to F^*_{k+1}. This yields a point $y_{k+1} \in Y$ such that

$$|d(y_{k+1}, z) - e^*(y^*_{k+1}, z)| < 2^{-k-1}\delta$$

for all $z \in F_k$. By definition of e^*, we have

$$|d(y_{k+1}, x_i) - d^*(y^*_{k+1}, z)| < 2^{-k-1}\delta$$

for $1 \leq i \leq n$, and (+ + +) yields

$$d(y_{k+1}, y_k) < e^*(y^*_{k+1}, y_k) + \varepsilon \leq 2^{-k-1}\delta + 2^{-k-1}\delta = 2^{-k}\delta$$

as required. $\square$

Lecture 4: Trees

Let A be a set. The set of all finite sequences over A is denoted by $A^{<\mathbb{N}}$.

Definition 4.1: A **tree** on A is a set $T \subseteq A^{<\mathbb{N}}$ that is *closed under prefixes*, that is

$$\forall \sigma, \tau \, [\tau \in T \,\&\, \sigma \subseteq \tau \;\Rightarrow\; \sigma \in T]$$

We call the elements of T *nodes*.

A sequence $\alpha \in A^{\mathbb{N}}$ is a **infinite path through** or **infinite branch of** T if for all n, $\alpha \restriction_n = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in T$. We denote the set of infinite paths through T by $[T]$.

An important criterion for a tree to have infinite paths is the following.

Theorem 4.2 (König's Lemma)**:** *If T is* finite branching*, i.e. each node has at most finitely many immediate extensions, then*

$$T \textit{ infinite} \quad \Rightarrow \quad T \textit{ has an infinite path}.$$

Proof sketch. We construct an infinite path inductively. Let T_σ denote the tree "above" σ, i.e. $T_\sigma = \{\tau \in A^{<\mathbb{N}} \colon \sigma^\frown\tau \in T\}$. If T is finite branching, by the *pigeonhole principle*, at least one of the sets T_σ for $|\sigma| = 1$ must be infinite. Pick such a σ and let $\alpha \restriction_1 = \sigma$. Repeat the argument for $T = T_\sigma$ and continue inductively. This yields a sequence $\alpha \in [T]$. $\square$

If $[T] = \emptyset$, we call T **well-founded**. The motivation behind this is that T is well-founded if and only if the *inverse prefix* relation

$$\sigma \preceq \tau \quad :\Longleftrightarrow \quad \sigma \supseteq \tau$$

is well-founded, i.e. it does not have an infinite descending chain.

If $T \neq \emptyset$ is well-founded, we can assign T an ordinal number, its **rank** $\rho(T)$.

- If σ is a *terminal node*, i.e. σ has no extensions in T, then let $\rho_T(\sigma) = 0$.
- If σ is not terminal, and $\rho_T(\tau)$ has been defined for all $\tau \supset \sigma$, we set $\rho_T(\sigma) = \sup\{\rho_T(\tau) + 1 \colon \tau \in T, \tau \supset \sigma\}$.
- Finally, set $\rho(T) = \sup\{\rho_T(\sigma) + 1 \colon \sigma \in T\} = \rho_T(\langle\varnothing\rangle) + 1$, where $\langle\varnothing\rangle$ denotes the empty string.

Orderings on trees

If A itself is linearly ordered, we can extend the inverse prefix ordering to a total ordering on $A^{<\mathbb{N}}$ So suppose $\leq$ is a linear ordering of A. The (partial) **lexicographical ordering** $\leq_{\text{lex}}$ of $A^{<\mathbb{N}}$ is defined as

$$\sigma \leq_{\text{lex}} \tau \quad \text{iff} \quad \sigma = \tau \ \text{ or } \ \exists i < \min\{|\sigma|, |\tau|\}, \ [(\forall j < i)\, \sigma_j = \tau_j \ \&\ \sigma_i < \tau_i]$$

This ordering extends to $A^{\mathbb{N}}$ in a natural way.

Proposition 4.3: *If $\leq$ is a well-ordering of A and T is a tree on A with $[T] \neq \emptyset$, then $[T]$ has a $\leq_{\text{lex}}$-minimal element, the* ***leftmost branch****.*

Proof. We *prune* the tree T by deleting any node that is not on an infinite branch. This yields a subtree $T' \subseteq T$ with $[T'] = [T]$. Let $T'_n = \{\sigma \in T' : |\sigma| = n\}$. Since $\leq$ is a well-ordering on A, T'_1 must have a $\leq_{\text{lex}}$-least element. Denote it by $\alpha \mid_1$. Since T' is pruned, $\alpha \mid_1$ must have an extension in T, and we can repeat the argument to obtain $\alpha \mid_2$. Continuing inductively, we define an infinite path α through T', and it is straightforward to check that α is a $\leq_{\text{lex}}$-minimal element of $[T']$ and hence of $[T]$. □

We can combine the $\leq_{\text{lex}}$-ordering with the inverse prefix order to obtain a linear ordering of $A^{<\mathbb{N}}$. This ordering has the nice property that if A is well-ordered and T is well-founded, then the ordering restricted to T is a well-ordering.

Definition 4.4: The **Kleene-Brouwer ordering** $\leq_{\text{KB}}$ of $A^{<\mathbb{N}}$ is defined as follows.

$$\sigma \leq_{\text{KB}} \tau \quad \text{iff} \quad \sigma \supseteq \tau \ \text{ or } \ \sigma \leq_{\text{lex}} \tau$$

This means σ is smaller than τ if it is a proper extension of τ or "to the left" of τ.

We now have

Proposition 4.5: *Assume $(A, \leq)$ is a well-ordered set. Then for any tree T on A,*

$$T \text{ is well-founded} \quad \Longleftrightarrow \quad \leq_{\text{KB}} \text{ restricted to } T \text{ is a well-ordering.}$$

Proof. Suppose T is not well-founded. Let $\alpha \in [T]$. Then $\alpha \mid_0, \alpha \mid_1, \dots$ is an infinite descending sequence with respect to $\leq_{\text{KB}}$.

Conversely, suppose $\sigma_0 >_{\text{KB}} \sigma_1 >_{\text{KB}} \dots$ is an infinite descending sequence on T. Then $\sigma_1(0) \geq \sigma_2(0) \geq \dots$ as a sequence in A. Since A is well-ordered, this sequence must eventually be constant, say $\sigma_n(0) = a_0$ for all $n \geq n_0$. Since the σ_n are descending, by the definition of $\leq_{\text{KB}}$ it follows that $|\sigma_n| \geq 2$ for $n > n_0$. Hence we can consider the sequence $\sigma_{n_0+1}(1) \geq \sigma_{n_0+2}(1) \geq \dots$ in A. Again, this must be constant $= a_1$ eventually. Inductively, we obtain a sequence $\alpha = (a_0, a_1, a_2, \dots) \in [T]$, i.e. T is not well-founded. $\square$

Note however that the order type of a well-founded tree under $\leq_{\text{KB}}$ is not the same as its rank $\rho(T)$.

Of course we can also define an ordering on $A^{<\mathbb{N}}$ via an injective mapping from $A^{<\mathbb{N}}$ to some linearly ordered set A. We will use this repeatedly for the case $A = \mathbb{N}$ and $A = \{0, 1\}$.

For $A = \mathbb{N}$, we can use the standard coding mapping

$$\pi : (a_0, a_1, \dots, a_n) \mapsto p_0^{a_0+1} p_1^{a_1} \cdots p_n^{a_n},$$

where p_k is the kth prime number. This embeds $\mathbb{N}^{<\mathbb{N}}$ into $\mathbb{N}$, and we can well-order $\mathbb{N}^{<\mathbb{N}}$ by letting $\sigma < \tau$ if and only if $\pi(\sigma) < \pi(\tau)$.

For $A = \{0, 1\}$ we set

$$\pi : (b_0, b_1, \dots, b_n) \mapsto \sum_{i=0}^{n} 2^{b_i}.$$

These two embedding allows us henceforth to see *trees as subsets of the natural numbers*. If we optimize the coding suitably, we can make it onto, and henceforth also assume that every subset of $\mathbb{N}$ codes a tree (on $\{0, 1\}$ or $\mathbb{N}$, depending on the circumstances). This will be an important component in exploring the relation between *topological and arithmetical complexity*.

Trees and closed sets

Let A be a set with the discrete topology. Consider $A^{\mathbb{N}}$ with the product topology defined in Lecture 2.

Proposition 4.6: *A set $F \subseteq A^{\mathbb{N}}$ is closed if and only if there exists a tree T on A such that $F = [T]$.*

Proof. Suppose F is closed. Let

$$T_F = \{\sigma \in A^{<\mathbb{N}} : \sigma \subset \alpha \text{ for some } \alpha \in F\}.$$

Then clearly $F \subset [T_F]$. Suppose $\alpha \in [T_F]$. This means for any n, $\alpha\,|_n \in T_F$, which implies that there exists $\beta_n \in F$ such that $\alpha_n \subset \beta_n$. The sequence (β_n) converges to α, and since F is closed, $\alpha \in F$.

For the other direction, suppose $F = [T]$. Let $\alpha \in A^{\mathbb{N}} \setminus F$. Then there exists an n such that $\alpha\,|_n \notin T$. Since a tree is closed under prefixes, since implies that no extension of $\alpha\,|_n$ can be in T. This in turn implies $N_{\alpha|_n} \subseteq A^{\mathbb{N}} \setminus F$, and hence $A^{\mathbb{N}} \setminus F$ is open. □

Trees and continuous mappings

Let $f : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be continuous. We define a mapping $\varphi : A^{<\mathbb{N}} \to A^{<\mathbb{N}}$ by setting

$$\varphi(\sigma) = \text{ the longest } \tau \text{ such that } N_\sigma \subseteq f^{-1}(N_\tau).$$

This mapping has the following properties:

(1) It is *monotone*, i.e. $\sigma \subseteq \tau$ implies $\varphi(\sigma) \subseteq \varphi(\tau)$.

(2) For any $\alpha \in A^{\mathbb{N}}$ we have $\lim_n |\varphi(\alpha\,|_n)| = \infty$. This follows directly from the continuity of f: For any neighborhood N_τ of $f(\alpha)$ there exists a neighborhood N_σ of α such that $f(N_\sigma) \subseteq N_\tau$. But τ has to be of the form $\tau = f(\alpha)\,|_m$, and σ of the form $\alpha\,|_n$. Hence for any m there must exist an n such that $\varphi(\alpha\,|_n) \supseteq f(\alpha)\,|_m$.

On the other hand, if a function $\varphi : A^{<\mathbb{N}} \to A^{<\mathbb{N}}$ satisfies (1) and (2), it induces a function $\varphi^* : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by letting

$$\varphi^*(\alpha) = \lim_n \varphi(\alpha\,|_n) = \text{ the unique sequence extending all } \varphi(\alpha\,|_n).$$

This φ^* is indeed continuous: The preimage of N_τ under φ^* is given by

$$(\varphi^*)^{-1}(N_\tau) = \bigcup \{N_\sigma : \varphi(\sigma) \supseteq \tau\},$$

which is an open set.

We have shown

Proposition 4.7: *A mapping $f : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is continuous if and only if there exists a mapping φ satisfying (1) and (2) such that $f = \varphi^*$.*

Again, note that we can completely describe a topological concept, continuity, through a relation between finite strings.

Lecture 5: Borel Sets

Topologically, the Borel sets in a topological space are the σ-algebra generated by the open sets. One can build up the Borel sets from the open sets by iterating the operations of complementation and taking countable unions. This generates sets that are more and more complicated, which is refelcted in the *Borel hierarchy*. The complexity is reflected on the logical side by the number of quantifier changes needed to define the set. There is a close connection between the arithmetical and the Borel hierarchy.

Definition 5.1: Let X be a set. A σ**-algebra** $\mathcal{S}$ on X is a collection of subsets of X such that $\mathcal{S}$ is closed under complements and countable unions, that is

- if $A \in \mathcal{S}$, then $X \setminus A \in \mathcal{S}$, and
- if $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{S}$, then $\bigcup_n A_n \in \mathcal{S}$,

If the enveloping space X is clear, we use $\neg A$ to denote the complement of A in X.

It is easy to derive that a σ-algebra is also closed under the following set-theoretic operations:

- *Countable intersections* – we have $\bigcap A_n = \neg \bigcup_n \neg A_n$.
- *Differences* – we have $A \setminus B = A \cap \neg B$.
- *Symmetric differences* – we have $A \triangle B = (A \cap \neg B) \cup (\neg A \cap B)$.

Definition 5.2: Let $(X, \mathcal{O})$ be a topological space. The collection of **Borel sets** in X is the smallest σ-algebra containing the open sets in $\mathcal{O}$.

One, of course, has to make sure that this collection actually exists. For this, note that the intersection of any collection of σ-algebras is again a σ-algebra, so the Borel sets are just the intersection of all σ-algebras containing $\mathcal{O}$. (Note the the full power set of X is such a σ-algebras, so we are not taking an empty intersection.)

This definition of Borel sets is rather *"external"*. It does not give us any idea what Borel sets *look like*. One can arrive at the family of Borel sets also through a construction from "within". This reveals more structure and gives rise to the *Borel hierarchy*.

The Borel hierarchy

We will restrict ourselves from now on to Polish spaces, to ensure that every closed set is a countable intersection of open sets (see exercises).

To generate the Borel sets, we start with the open sets. By closing under complements, we obtain the closed sets. We also have to close under countable unions. The open sets are already closed under this operation, but the closed sets are not. Countable unions of closed sets are classically known as F_σ sets. Their complements, i.e. countable intersections of open sets, are the G_δ *sets*. We can continue this way and form the $F_{\sigma\delta}$ sets – countable intersections of F_σ sets – the $G_{\delta\sigma}$ sets – countable unions of G_δ sets – and so on. It is obvious that the $\sigma\delta$-notation soon becomes rather impractical, and hence we replace it by something much more convenient, and much more *suggestive*, as we will see later.

Definition 5.3 (Borel sets of finite order)**:** Let X be a Polish space. We inductively define the following collection of subsets of X.

$$\begin{aligned}
&\boldsymbol{\Sigma}^0_1(X) = \{U : U \subseteq X \text{ open }\}\\
&\boldsymbol{\Pi}^0_n(X) = \{\neg A : A \in \boldsymbol{\Sigma}^0_n(X)\} = \neg\,\boldsymbol{\Sigma}^0_n(X)\\
&\boldsymbol{\Sigma}^0_{n+1}(X) = \{\bigcup_k A_k : A_k \in \boldsymbol{\Pi}^0_n(X)\}
\end{aligned}$$

Hence the open sets are precisely the sets in $\boldsymbol{\Sigma}^0_1$, the closed sets are the sets in $\boldsymbol{\Pi}^0_1$, the F_σ sets from the class $\boldsymbol{\Sigma}^0_2$ etc. If it is clear what the underlying space X is, we drop the reference to it and simply write $\boldsymbol{\Sigma}^0_n$ and $\boldsymbol{\Pi}^0_n$. Besides, we will say that a set $A \subseteq X$ *is* (or *is not*) $\boldsymbol{\Sigma}^0_n$ or $\boldsymbol{\Pi}^0_n$, respectively.

Does the collection of all $\boldsymbol{\Sigma}^0_n$ and $\boldsymbol{\Pi}^0_n$ exhaust the Borel sets of X? We will see that the answer is no. We have to extend our inductive construction into the transfinite and consider classes $\boldsymbol{\Sigma}^0_\xi$, where ξ is a countable infinite ordinal.

The Borel sets of finite order

We fix a Polish space X. We want to establish the basic relationships between the different classes $\boldsymbol{\Sigma}^0_n$ and $\boldsymbol{\Pi}^0_m$ for X.

It is clear that $\boldsymbol{\Sigma}^0_1 \nsubseteq \boldsymbol{\Pi}^0_1$ and $\boldsymbol{\Pi}^0_1 \nsubseteq \boldsymbol{\Sigma}^0_1$. Furthermore, it follows from the definitions that $\boldsymbol{\Pi}^0_n \subseteq \boldsymbol{\Sigma}^0_{n+1}$ and $\boldsymbol{\Sigma}^0_n \subseteq \boldsymbol{\Pi}_{n+1}$.

Lemma 5.4: *In a Polish metric space (X,d), every open set is an F_σ set.*

Proof. Let $D = \{x_1, x_2, \dots\} \subseteq X$ be a countable dense subset, and assume $U \subseteq X$ is open. For any $\varepsilon > 0$, if $\delta < \varepsilon$, then $\overline{U_\delta(x)} \subseteq U_\varepsilon(x)$ for any $x \in X$. Let $x_{i(1)}, x_{i(2)}, \dots$ and $\varepsilon_1, \varepsilon_2, \dots$ be such that

$$U = \bigcup_n U_{\varepsilon_n}(x_{i(n)}).$$

For each $n \geq 1$, let $(\delta_k^{(n)})$ be such that $\delta_k^{(n)} < \delta_{k+1}^{(n)} < \cdots < \varepsilon_i$, and $\delta_k^{(n)} \to \varepsilon_n$. Then

$$U = \bigcup_k \bigcup_n \overline{U_{\delta_k^{(n)}}(x_{i(n)})}.$$

The set on the right hand side is a countable union of closed sets. $\square$

Corollary 5.5: $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$ *and* $\mathbf{\Pi}^0_1 \subseteq \mathbf{\Pi}^0_2$.

The second statement follows by passing to complements: If F is closed,

$$F = \neg\neg F = \neg \bigcup F_n = \bigcup \neg F_n,$$

where the F_n are closed.

There are also sets that can be both $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$, but neither $\mathbf{\Sigma}^0_1$ nor $\mathbf{\Pi}^0_1$. For example, consider the half-open interval $[0,1)$.

$$[0,1) = \bigcup_n [1, 1-1/n] = \bigcap_m (-1/n, 1).$$

Therefore, it makes sense to define the **hybrid classes**

$$\mathbf{\Delta}^0_n = \mathbf{\Sigma}^0_n \cap \mathbf{\Pi}^0_n.$$

Using induction, we can extend the inclusions in a straightforward way to higher n.

Theorem 5.6 (Weak Hierarchy Theorem)**:**

$$
\begin{array}{ccccccccccccc}
 & & \mathbf{\Sigma}^0_1 & & & & \mathbf{\Sigma}^0_2 & & & & \mathbf{\Sigma}^0_3 & & \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\mathbf{\Delta}^0_1 & & & & \mathbf{\Delta}^0_2 & & & & \mathbf{\Delta}^0_3 & & & & \cdots \\
 & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \mathbf{\Pi}^0_1 & & & & \mathbf{\Pi}^0_2 & & & & \mathbf{\Pi}^0_3 & &
\end{array}
$$

We also want show that the inclusions are proper. For the first two levels, this can be done by explicit counterexamples. Any countable set is in $\mathbf{\Sigma}^0_2$, since a singleton set is closed, and a countable set is a countable union of singletons. However, there are countable sets that are neither open nor closed, e.g. $\{1/n : n \geq 1\}$. The complement is consequently a $\mathbf{\Pi}^0_2$ set that is neither open nor closed. Furthermore, the rationals $\mathbb{Q}$ give an example of a $\mathbf{\Sigma}^0_2$ set that is not $\mathbf{\Pi}^0_2$. This will be shown later using the concept of *Baire category*.

It is much harder to find specific examples for the higher levels, e.g. a $\mathbf{\Sigma}^0_5$ set that is not $\mathbf{\Sigma}^0_4$. This separation will be much facilitated by the introduction of a logical/definability framework for the Borel sets. Therefore, we defer the proof for a while.

Examples of Borel sets – Continuity points of functions

Theorem 5.7 (Young)**:** *Let $f : X \to Y$ be a mapping between Polish spaces. Then*

$$C_f = \{x : f \text{ is continuous at } x\}$$

is a $\mathbf{\Pi}^0_2$ (i.e. G_δ) set.

Proof. It is not hard to see that f is continuous at a if and only if for any $\varepsilon > 0$,

$$\exists \delta > 0 \, \forall x, y \, [x, y \in U_\delta(a) \Rightarrow d(f(x), f(y)) < \varepsilon]. \tag{$*$}$$

Given $\varepsilon > 0$, let

$$C_\varepsilon = \{a : \ (*) \text{ holds at } a \text{ for } \varepsilon\}.$$

We claim that C_ε is open. Suppose $a \in C_\varepsilon$. Choose a suitable δ that witnesses that $a \in C_\varepsilon$. We show $U_\delta(a) \subseteq C_\varepsilon$. Let $b \in U_\delta(a)$. Choose δ^* so that $U_{\delta^*}(b) \subseteq U_\delta(a)$. Then

$$x, y \in U_{\delta^*}(b) \Rightarrow x, y \in U_\delta(a) \Rightarrow d(f(x), f(y)) < \varepsilon.$$

Notice further that $\varepsilon > \varepsilon^*$ implies $C_\varepsilon \supseteq C_{\varepsilon^*}$. Hence we can represent C_f as

$$C_f = \bigcap_{n \in \mathbb{N}} C_{1/n},$$

a countable intersection of open sets. $\square$

The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \text{ irrational}, \\ 0 & x = 0, \\ 1/q & x = p/q,\ p \in \mathbb{Z},\ q \in \mathbb{Z}^{>0},\ p, q \text{ relatively prime} \end{cases}$$

is a function that is continuous at every irrational, discontinuous at every rational number. As noted above, the rationals are a $\mathbf{\Sigma}^0_2$ set that is not $\mathbf{\Pi}^0_2$. Hence there cannot exist a function $g : \mathbb{R} \to \mathbb{R}$ that is discontinuous at exactly the irrationals.

We finish this lecture by showing that Young's Theorem can be reversed.

Theorem 5.8: *Given a $\mathbf{\Pi}^0_2$ subset A of a perfect Polish space X, there exists a mapping $f : X \to \mathbb{R}$ such that f is continuous at every point in A, and discontinuous at every other point, i.e. $C_f = A$.*

Proof. Fix a countable dense subset $D \subseteq X$. We first deal with the easier case that A is open. Let

$$f(x) = \begin{cases} 0 & x \in A \text{ or } x \in \neg\, \overline{A} \cap D, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that f is continuous on A. Now assume $x \notin A$. If $x \notin \overline{A}$, then there exists $U_\varepsilon(x) \subseteq \neg\, \overline{A}$. Any $U_{\varepsilon^*}(x) \subseteq U_\varepsilon(x)$ contains points from both D and $\neg\, D$, so it is clear that f is not continuous at x. Finally, let $x \in \neg A \setminus A$. Then $f(x) = 1$, but points of A are arbitrarily close, where f takes value 0.

Now we extend this approach to general $\mathbf{\Pi}^0_2$ sets. Suppose

$$A = \bigcap_n G_n, \qquad G_n \text{ open.}$$

By replacing G_n with $G^*_n = G_1 \cap \cdots \cap G_n$, we can assume that

$$X = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots$$

The idea is to define f_n as above for each G_n and then "amalgamate" the f_n is a suitable way. Assume for each n, $f_n : X \to \mathbb{R}$ is defined as above such that $C_{f_n} = G_n$. Let (b_n) be a sequence of positive real numbers such that for all n,

$$b_n > \sum_{k>n} b_k,$$

for example, $b_n = 1/n!$. We now form the series

$$f(x) = \sum_n b_n f_n(x).$$

Since $|f_n(x)| \leq 1$, $|f(x)| \leq \sum_n b_n < \infty$. Furthermore, (f_n) converges uniformly to f, for

$$|f(x) - f_n(x)| \leq \sum_{k>n} b_k < b_n,$$

and the last bound is independent of x and converges to 0.

It follows by uniform convergence that if each f_n is continuous at x, f is continuous on x, too. Hence f is continuous on A.

Now assume $x \notin A$. Then there exist n such that $x \in G_n \setminus G_{n+1}$. Hence

$$f_0(x) = \cdots = f_n(x) = 0.$$

Again, we distinguish two cases. First, assume $x \notin \overline{G_{n+1}}$. Then there exists $\delta > 0$ such that $U_\delta(x) \subseteq \neg G_{n+1}$. This also implies $U_\delta(x) \subseteq \neg G_k$ for any $k \geq n+1$. Besides, since G_n is open, we can chose δ sufficiently small so that $U_\delta(x) \subseteq G_n$. For $y \in \neg D \cap U_\delta(x)$ we have $f_k(y) = 1$ for all $k \geq n+1$, and hence $f(y) = \sum_{k>n} b_k f_k(y) > 0$. On the other hand, if $y \in D \cap U_\delta(x)$, then $f_k(y) = 0$ for all $k \geq n+1$, and also $f_0(y) = \cdots = f_n(y) = 0$, since $y \in G_n$, and thus $f(y) = 0$. Hence there are points arbitrarily close to x whose f-values differ by a constant lower bound, which implies f is not continuous in x.

Finally, suppose $x \in \overline{G_{n+1}}$. Then $f_{n+1}(x) = 1$ and hence $f(x) \geq b_{n+1} > 0$. On the other hand, for any $y \in G_{n+1}$, $f(y) \leq \sum_{k>n+1} b_k < b_{n+1} = f(x)$. That is, there are points arbitrarily close to x whose f-value differs from $f(x)$ by a constant lower bound. Hence f is discontinuous at x in this case, too. $\square$

Lecture 6: Borel Sets as Clopen Sets

In this lecture we will learn that the Borel sets have the perfect subset property, which we already saw holds for closed subsets of Polish spaces.

The proof changes the underlying topology so that all Borel sets become clopen, and hence we can apply the Cantor-Bendixson Theorem 2.5.

We start by showing that topologically simple subspaces of Polish spaces are again Polish

Proposition 6.1: *If Y is an open or closed subset of a Polish space X, then Y is Polish, too (with respect to the subspace topology).*

Proof. Clearly subspaces of separable spaces are separable.

The statement for closed sets follows easily, since closed subsets of complete metric spaces are complete.

In case Y is open, suppose d is a compatible metric on X such that (X,d) is complete. Y may not be complete with respect to d, so we have to change the metric, but be careful not to change the induced topology.

First, replace the metric d by $\overline{d}$, given as

$$\overline{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

This is again a metric, and it induces the same topology, for one can show that the identity mapping is a homeomorphism of (X,d) and $(X,\overline{d})$.

Now define

$$d_Y(x,y) = \overline{d}(x,y) + \left| \frac{1}{\overline{d}(x,X\setminus Y)} - \frac{1}{\overline{d}(y,X\setminus Y)} \right|,$$

where $\overline{d}(x,Z) = \inf\{d(x,z) \colon z \in Z\}$. With some effort, one can show that this is again a metric.

To show it is compatible with the subspace topology, assume $x_n \to z$ in (Y,d_Y). Since $d \le d_y$, we have $d(x_n,z) \to 0$, and hence $x_n \to z$ in (Y,d).

On the other hand, if $d(x_n, z) \to 0$ in Y, it follows that $\overline{d}(x_n, z) \to 0$. Furthermore, using the triangle-inequality, we can show that the sequence

$$\frac{1}{\overline{d}(x_n, X \setminus Y)} - \frac{1}{\overline{d}(z, X \setminus Y)}$$

also goes to zero. Hence $x_n \to z$ in (Y, d_Y).

Finally, assume (x_n) is Cauchy in (Y, d_Y). Since $d \le d_Y$, it is also Cauchy in (X, d), hence the exists $x \in X$ with $x_n \to x$. Using the triangle-inequality, one can further show that the sequence

$$\frac{1}{\overline{d}(x_n, X \setminus Y)}$$

is Cauchy in $\mathbb{R}$. Hence there exists $r \in \mathbb{R}$ such that

$$r = \lim_n \frac{1}{\overline{d}(x_n, X \setminus Y)}.$$

Now, since $\overline{d} < 1$, r cannot be 0, and hence

$$\frac{1}{\overline{d}(x_n, X \setminus Y)}$$

is bounded away from 0, which implies (triangle-inequality) that $d(x, X \setminus Y)$ is bounded away from 0, too. But this means $x \in Y$, hence (x_n) converges in (Y, d_Y). $\square$

One can strengthen this result to G_δ sets, in fact, the Polish subspaces of Polish spaces are *precisely* the G_δ subsets.

Theorem 6.2: *A subset of a Polish space is Polish (with the subspace topology) if and only if it is* $\mathbf{\Pi}^0_2$.

For a proof see **(author?)** [Kec95].

Next we show that the topology can be refined to make closed subsets clopen.

Lemma 6.3: *If X is a Polish space with topology $\mathcal{O}$, and $F \subseteq X$ is closed, then there exists a finer topology $\mathcal{O}' \supseteq \mathcal{O}$ such that $\mathcal{O}$ and $\mathcal{O}'$ give rise to the same class of Borel sets in X, and F is clopen with respect to $\mathcal{O}'$.*

Proof. By Proposition 6.1, F and $X \setminus F$ are Polish spaces with compatible metrics d_F and $d_{X\setminus F}$, respectively. Wlog $d_F, d_{X\setminus F} < 1$. We form the *disjoint union* of the *spaces* F and $X \setminus F$: This is the set $X = F \sqcup X \setminus F$ with the following topology $\mathcal{O}'$. $U \subseteq F \sqcup X \setminus F$ is in $\mathcal{O}'$ if and only if $U \cap F$ is open (in F) and $U \cap X \setminus F$ is open (in $X \setminus F$).

The disjoint union is Polish, as witnessed by the following metric.

$$d_{\sqcup}(x,y) = \begin{cases} d_F(x,y) & \text{if } x,y \in F, \\ d_{X\setminus F}(x,y) & \text{if } x,y \in X \setminus F, \\ 2 & \text{otherwise.} \end{cases}$$

It is straightforward to check that d is compatible with $\mathcal{O}'$. Furthermore, let (x_n) be Cauchy in $(X, d_{\sqcup})$. Then the x_n are completely in F or in $X \setminus F$ from some point on, and hence (x_n) converges.

Under the disjoint union topology, F is is clopen. Moreover, an open set in this topology is a disjoint union of an open set in $X \setminus F$, which also open the original topology $\mathcal{O}$, and an intersection of an open set from $\mathcal{O}$ with F. Such sets are are Borel in $(X, \mathcal{O})$, hence $(X, \mathcal{O})$ and $(X, \mathcal{O}')$ have the same Borel sets. □

Theorem 6.4: *Let X be a Polish space with topology $\mathcal{O}$, and suppose $B \subseteq X$ is Borel. Then there exists a finer topology $\mathcal{O}' \supseteq \mathcal{O}$ such that $\mathcal{O}$ and $\mathcal{O}'$ give rise to the same class of Borel sets in X, and F is clopen with respect to $\mathcal{O}'$.*

Proof. Let $\mathcal{S}$ be the family of all subsets A of X for which a finer topology exists that has the same Borel sets as $\mathcal{O}$ and in which A is open.

We will show that $\mathcal{S}$ is a σ-algebra, which by the previous Lemma contains the closed sets. Hence $\mathcal{S}$ must contain all Borel sets, and we are done.

$\mathcal{S}$ is clearly closed under complements, since the complement of a clopen set is clopen in any topology.

So assume now that $\{A_n\}$ is a countable family of sets in $\mathcal{S}$. Let $\mathcal{O}_n$ be a topology on X that makes A_n clopen and does not introduce new Borel sets.

Let $\mathcal{O}_\infty$ be the topology generated by $\bigcup_n \mathcal{O}_n$. Then $\bigcup_n A_n$ is open in $(X, \mathcal{O}_\infty)$, and we can apply Lemma 6.3. For this to work, however, we have to show that $(X, \mathcal{O}_\infty)$ is Polish and does not introduce any new Borel sets.

We know that the *product space* $\prod(X, \mathcal{O}_n)$ is Polish. Consider the mapping $\varphi : X \to \prod_n X$

$$x \mapsto (x, x, x, \ldots).$$

Observe that φ is a continuous mapping between $(X, \mathcal{O}_\infty)$ and $\prod_n X$. The preimage of a basic open set $U_1 \times U_2 \times \cdots \times U_n \times X \times X \times \cdots$ under φ is just the intersection of the U_i. Furthermore, φ is clearly one-to-one, and the inverse mapping between $\varphi(X)$ and X is continuous, too.

If we can show that $\varphi(X)$ is closed in $\prod_n X$, we know it is Polish as a closed subset of a Polish space, and since $(X, \mathcal{O}_\infty)$ is homeomorphic to $\varphi(X)$, we can conclude it is Polish.

To see that $\varphi(X)$ is closed in $\prod_n X$, let $(y_1, y_2, y_3, \dots) \in \neg\varphi(X)$. Then there exist $i < j$ such that $y_i \neq y_j$. Since $(X, \mathcal{O})$ is Polish, we can pick U, V open, disjoint such that $y_i \in U$, $y_j \in V$. Since each $\mathcal{O}_n$ refines $\mathcal{O}$, U is open in $\mathcal{O}_i$, and V is open in $\mathcal{O}_j$. Therefore,

$$X_1 \times X_2 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times \cdots \times X_{j_1} \times V \times X_{j+1} \times X_{j+2} \times \cdots$$

where $X_k = X$ for $k \neq i, j$, is an open neighborhood of $(y_1, y_2, y_3, \dots)$ completely contained in $\neg\varphi(X)$.

Finally, too see that the Borel sets of $(X, \mathcal{O}_\infty)$ are the same as the ones of $(X, \mathcal{O})$, for each n, let $\{U_i^{(n)}\}_{i \in \mathbb{N}}$ be a basis for $\mathcal{O}_n$. By assumption, all sets in $\mathcal{O}_n$ are Borel sets of $(X, \mathcal{O})$. The set $\{U_i^{(n)}\}_{i,n \in \mathbb{N}}$ is a subbasis for $\mathcal{O}_\infty$. This means that any open set in $(X, \mathcal{O}_\infty)$ is a countable union of finite intersections of the $U_i^{(n)}$. Since every $U_i^{(n)}$ is Borel in $(X, \mathcal{O})$, this means that any open set in $\mathcal{O}_\infty$ is Borel in $(X, \mathcal{O})$. Since the Borel sets are closed under complementation and countable unions, this in turn implies that very Borel set of $(X, \mathcal{O}_\infty)$ is already Borel in $(X, \mathcal{O})$.

□

Corollary 6.5 (Perfect subset property for Borel sets): *In a Polish space, every uncountable Borel set has a perfect subset.*

Proof. Let $(X, \mathcal{O})$ be Polish, and assume $B \subseteq X$ is Borel. We can choose a finer topology $\mathcal{O}' \supseteq \mathcal{O}$ so that B becomes clopen, but the Borel sets stay the same. B is Polish with respect to the subspace topology $\mathcal{O}'|_B$

By Theorem 2.4, there exists a continuous injection f from $2^{\mathbb{N}}$ (with respect to the standard topology) into $(B, \mathcal{O}'|_B)$. Since $2^{\mathbb{N}}$ is compact, $f(2^{\mathbb{N}})$ is closed in $(B, \mathcal{O}'|_B)$. Since $\mathcal{O}' \supseteq \mathcal{O}$, every closed set in $(B, \mathcal{O}'|_B)$ is also closed in $\mathcal{O}$. Likewise, f is continuous between $2^{\mathbb{N}}$ and $(B, \mathcal{O}|_B)$, too. Therefore, $f(2^{\mathbb{N}})$ has no isolated points with respect to $\mathcal{O}$. It follows that $f(2^{\mathbb{N}})$ is perfect with respect to $\mathcal{O}$. □

Lecture 7: Measure and Category

The Borel hierarchy classifies subsets of the reals by their topological complexity. Another approach is to classify them by "size".

Filters and Ideals

The most common measure of size is, of course, cardinality. In the presence of uncountable sets (like in a perfect Polish space), the usual division is between countable and uncountable sets. The smallness of the countable sets is reflected, in particular, by two properties: A subset of a countable set is countable, and countable unions of countable set are countable. These characteristics are shared with other notions of smallness, two of which we will encounter in this lecture.

Definition 7.1: A non-empty family $\mathcal{I} \subseteq \mathcal{P}(X)$ of subsets of a given set X is an **ideal** if

(I1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$,

(II2) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

If we have closure even under *countable* unions, we speak simply of a *σ-ideal*. For example, while the countable sets in $\mathbb{R}$ form a σ-ideal, the finite subsets only form an ideal.

Another example of ideals are the so-called **principal ideals**. These are ideals of the form

$$\langle Z \rangle = \{A \colon A \subseteq Z\}$$

for a fixed $Z \subseteq X$.

The *dual notion* to an ideal is that of a filter. It reflects that the sets in a filter share some *largeness property*.

Definition 7.2: A non-empty family $\mathcal{F} \subseteq \mathcal{P}(X)$ of subsets of a given set X is an **σ-filter** if

(F1) $A \in \mathcal{F}$ and $B \supseteq A$ implies $B \in \mathcal{F}$,

(F2) $A, B \in \mathcal{F}$ for all n implies $A \cap B \in \mathcal{F}$.

Again, closure under *countable* intersection yields *σ-filters*.

If $\mathcal{I}$ is a (σ-) ideal, then $\mathcal{F} = \{\neg A : A \in \mathcal{I}\}$ is a (σ-) filter. Hence the co-finite subsets of $\mathbb{R}$ form a filter, and the co-countable subsets form a σ-filter.

Note that the complement of a (σ-) ideal (in $\mathcal{P}(X)$) is not necessarily a (σ-) filter. This is true, however, for a special class of ideals/filters.

Definition 7.3: A non-empty family $\mathcal{I} \subseteq \mathcal{P}(X)$ is a **prime ideal** if it is an ideal for which

$$\text{for every } A \in X, \text{ either } A \in \mathcal{I} \text{ or } \neg A \in \mathcal{I}.$$

An **ultrafilter** is a filter whose complement in $\mathcal{P}(X)$ is a prime ideal.

In light of the small-/largeness motivation, prime ideals and ultrafilters provide a *complete* separation of X: Each set is either small or large.

Measures

Coarsely speaking, a measure assigns a size to a set in a way that reflects our basic geometric intuition about sizes: The size of the union of disjoint objects is the sum of their sizes. The question whether this can be done in a consistent way for *all* subsets of a given space is of fundamental importance and has motivated many questions in set theory.

The formally, a measure μ on X is a $[0,\infty]$-valued function defined on subsets of X that satisfies

(M1) $\mu(\emptyset) = 0$,

(M2) $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$, whenever the A_n are pairwise disjoint.

The question is, of course, which subsets of X can be assigned a measure. The condition (M2) suggests that this family is closed under countable unions. Furthermore, if $A \subseteq X$, then the equation $\mu(X) = \mu(A) + \mu(\neg A)$ suggests that $\neg A$ should be measurable, too. In other words, the sets who are assigned a measure form a σ-algebra.

Definition 7.4: A **measurable space** is a pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X. A **measure** on a measurable space $(X, \mathcal{S})$ is a function $\mu : \mathcal{S} \to [0,\infty]$ that satis fies (M1) and (M2) for any pairwise disjoint family $\{A_n\}$ in $\mathcal{S}$. If μ is a measure on $(X, \mathcal{S})$, then the triple $(X, \mathcal{S}, \mu)$ is called a **measure space**.

If we want the measure μ to reflect also some other basic intuition about geometric sizes, this often puts restrictions on the σ-algebra of measurable sets. For example, in $\mathbb{R}$ the measure of an interval should be its *length*. We will see later (when we discuss the Axiom of Choice) that it is impossible to assign every subset of $\mathbb{R}$ a measure, so that (M1) and (M2) are satisfied, and the measure of an interval is its length.

To have some control over what the σ-algebra of measurable sets should be, one can construct a measure more carefully, start with a measure on basic objects such as intervals or balls, and then extend it to larger classes of sets by approximation.

An essential component in this extension process is the concept of an *outer measure*.

Definition 7.5: An **outer measure** on a set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

(O1) $\mu^*(\emptyset) = 0$,

(O2) $A \subseteq B$ implies $\mu^*(A) \leq \mu^*(B)$,

(O3) $\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$, for any countable family $\{A_n\}$ is X.

An outer measure hence weakens the conditions of **additivity** (M2) to **subadditivity** (O3). This makes it possible to have non-trivial outer measures that are defined on *all* subsets of X.

The usefulness of outer measures lies in the fact that they can always be restricted to subset of $\mathcal{P}(X)$ on which they behave as measures.

Definition 7.6: Let μ^* be an outer measure on X. A set $A \subseteq X$ is μ^***-measurable** if

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A) \quad \text{for all } B \subseteq X.$$

This definition is a rather obscure. It is justified rather by its consequences than its intuitive appeal. Regarding the latter, suffice it to say here that outer measures may be rather far from being even *finitely* additive. The definition singles out those sets that split all other sets correctly, with regard to measure.

Proposition 7.7: *The class of μ^*-measurable sets forms a σ-algebra $\mathcal{M}$, and the restriction of μ^* to $\mathcal{M}$ is a measure.*

For a proof see for instance **(author?)** [Hal50].

The size of the σ-algebra of measurable sets depends, of course, on the outer measure μ^*. If μ^* is behaving rather pathetically, we cannot expect $\mathcal{M}$ to contain many sets.

Lebesgue measure

A standard way to obtain 'nice' outer measures is to start with a well-behaved function defined on a certain class of sets, and then approximate. The paradigm for this approach is the construction of **Lebesgue measure** on $\mathbb{R}$.

Definition 7.8: The **Lebesgue outer measure** λ^* of a set $A \subseteq \mathbb{R}$ is defined as

$$\lambda^*(A) = \inf\left\{\sum_n |b_n - a_n| : A \subseteq \bigcup_n (a_n, b_n)\right\}.$$

One can show that this indeed defines an outer measure. We call the λ^*-measurable sets **Lebesgue measurable**. One can verify that every open interval is Lebesgue measurable. It follows from Proposition 7.7 that every Borel set is Lebesgue measurable.

The construction of Lebesgue measure can be generalized and extended to other metric spaces, for example through the concept of *Hausdorff measures*.

All these measures are **Borel measures**, in the sense that the Borel measures are measurable. However, there measurable sets that are not Borel sets. The reason for this lies in the presence of *nullsets*, which are measure theoretically 'easy' (since they do not contribute any measure at all), but can be topologically quite complicated.

Nullsets

Let μ^* be an outer measure on X. If $\mu^*(A) = 0$, then A is called a μ^*-nullset.

Proposition 7.9: *Any μ^*-nullset is μ^*-measurable.*

Proof. Suppose $\mu^*(A) = 0$. Let $B \subseteq X$. Then, since μ^* is subadditive and monotone,

$$\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap \neg A) = \mu^*(B \cap \neg A) \leq \mu^*(B),$$

and therefore $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap \neg A)$. □

The next result confirms the intuition that nullsets are a notion of smallness.

Proposition 7.10: *The μ^*-nullsets form a σ-ideal.*

Proof. (I1) follows directly from monotonicity (O2). Countable additivity follows immediately from subadditivity (O3). $\square$

In case of Lebesgue measure, we can use Proposition 7.9 to further describe the Lebesgue measurable subsets of $\mathbb{R}$.

Proposition 7.11: *A set $A \subseteq \mathbb{R}$ is Lebesgue measurable if and only if it is the difference of a $\mathbf{\Pi}^0_2$ set and a nullset*

Proof. We first assume $\lambda^*(A) < \infty$. Let $G_n \subseteq \mathbb{R}$ be an open set such that $G_n \supseteq A$ and $\lambda^*(G_n) \leq \lambda^*(A) + 1/n$. The existence of such a G_n follows from the definition of λ^*, and the fact that every open set is the disjoint union of open intervals. Then $G = \bigcap_n G_n$ is $\mathbf{\Pi}^0_2$, $A \subseteq G$, and for all n,

$$\lambda^*(A) \leq \lambda^*(G) \leq \lambda^*(A) + 1/n$$

hence $\lambda^*(A) = \lambda^*(G)$. Hence for $N = G \setminus A$, since A is measurable,

$$\lambda^*(N) = \lambda^*(G) - \lambda^*(A) = 0 \quad \text{and} \quad A = G \setminus N.$$

If $\lambda^*(A) = \infty$, we set $A_m = A \cap [m, m+1)$ for $m \in \mathbb{Z}$. By monotonicity, each $\lambda^*(A_m)$ is finite. For each $m \in Integer$, $n \in \mathbb{N}$, pick $G_n^{(m)}$ open such that $\lambda^*(G_n^{(m)}) \leq \lambda^*(A) + 1/2^{n+2|m|+1}$. Then, with

$$\bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{Z}} G_n^{(m)},$$

$N = G \setminus A$ is the desired set.

For the other direction, note that the measurable sets form a σ-algebra which contains both the Borel sets and the nullsets. Hence any set that is the difference of a Borel set and a nullset is measurable, too. $\square$

One can also show that each Lebesgue measurable set can be written as a disjoint union of a $\mathbf{\Sigma}^0_2$ set and a nullset. Hence if a set is measurable, it differs from a (rather simple) Borel set only by a nullset.

We also obtain the following characterization of the σ-algebra of Lebesgue measurable sets.

Proposition 7.12: *The σ-algebra of Lebesgue measurable sets in $\mathbb{R}$ is the smallest σ-algebra containing the open sets and the nullsets.*

As mentioned before, there are Lebesgue measurable sets that are not Borel sets. We will eventually encounter such sets. The question which sets exactly are Lebesgue measurable was one of the major questions that drove the development of set theory, just like the question which uncountable sets have perfect subsets.

Baire category

The basic paradigm for smallness here is of topological nature. A set is small if it does not look anything like an open set, not even under closure. In the following, let X be a Polish space.

Definition 7.13: A set $A \subseteq X$ is **nowhere dense** if its complement contains an open, dense set.

That means for any open set $U \subseteq X$ we can find a subset $V \subseteq U$ such that $V \subseteq \neg A$. In other words, a nowhere dense set is "full of holes" (Oxtoby).

Examples of nowhere dense sets are all finite, or more generally, all discrete subsets of a perfect Polish space, i.e. sets all whose points are isolated. There are non-discrete nowhere dense sets, such as $\{0\} \cup \{1/n \colon n \in \mathbb{N}\}$ in $\mathbb{R}$, even uncountable ones, such as the middle-third Cantor set.

The nowhere dense sets form an ideal, but not a σ-ideal: Every singleton set is nowhere dense, but there are countable sets that are not, such as the rationals $\mathbb{Q}$ in $\mathbb{R}$.

To obtain a σ-ideal, we close the nowhere dense sets under countable unions.

Definition 7.14: A set $A \subseteq X$ is **meager** or of **first category** if it is the countable union of nowhere dense sets. Non-meager sets are also called sets of **second category**. Complements of meager sets are called **comeager** or **residual**.

The meager subsets of X form a σ-ideal. Examples of meager sets are all countable sets, but there are uncountable ones (Cantor set).

Baire category is often used in existence proofs: To show that a set with a certain property exists, one shows that the set of points not having the property. A famous example is Banach's proof of the existence of continuous, nowhere differentiable functions. For this to work, of course, we have to ensure that the complements of meagre sets are non-empty.

Theorem 7.15 (Baire Category Theorem)**:** *For any Polish space X, the following statements hold.*

(a) *For every meager set $M \subseteq X$, the complement $\neg M$ is dense in X.*

(b) *No open set is meager.*

(c) *If $\{D_n\}$ is a countable family of open, dense sets, then $\bigcap_n D_n$ is dense.*

Proof. (a) Assume $M = \bigcup_n N_n$, where each N_n is nowhere dense. Then $\neg M = \bigcap D_n$, where each D_n contains a dense, open set. Let $U \subseteq X$ be open. We construct a point $x \in U \cap \neg M$ by induction. We can find an open ball B_1 of radius < 1 such that $\overline{B_1} \subseteq U \cap D_1$, since D_1 contains a dense open set. In the next step, we use the same property of D_2 to find an open ball B_2 of radius $< 1/2$ whose closure is completely contained in $B_1 \cap D_2$. Continuing inductively, we obtain a nested sequence of balls B_n of radius $< 1/n$ such that $\overline{B_n} \subseteq B_{n-1} \cap D_n$. Let x_n be the center of B_n. Then (x_n) is a Cauchy sequence, so $x = \lim_n x_n$ exists in X. Since for any n, all but finitely many x_i are in B_n, we have $x \in \overline{B_n}$ for all n. Therefore, by construction

$$x \in \bigcap_n \overline{B_n} = \bigcap_n B_n \subseteq U \cap \bigcap_n D_n \subseteq U.$$

(b) follows immediately from (a), the proof of (c) is exactly the same as that for (a). In fact, the three statements are equivalent. □

As an application, we determine the exact location of $\mathbb{Q}$ in the Borel hierarchy of $\mathbb{R}$.

Corollary 7.16: $\mathbb{Q}$ *is not a* $\mathbf{\Pi}^0_2$ *set, hence a true* $\mathbf{\Sigma}^0_2$ *set.*

Proof. Note that $\mathbb{R}$ cannot be meager, by (b). Since $\mathbb{Q}$ is meager, $\mathbb{R} \setminus \mathbb{Q}$ cannot be meager either. If $\mathbb{Q}$ were a $\mathbf{\Pi}^0_2$ set, it would be the intersection of open, dense sets and hence its complement $\mathbb{R} \setminus \mathbb{Q}$ would be meager. □

We have seen that the measurable sets are precisely the ones that differ from a $\mathbf{\Pi}^0_2$ set by a nullset. We can introduce a similar concept for Baire category.

Definition 7.17: A set $B \subseteq X$ has the **Baire property** if there exists an open set G and a meager set M such that

$$B \triangle G = M.$$

The sets having the Baire property form a σ-algebra and hence include all Borel sets. Similar to measure, one has

Proposition 7.18: *The σ-algebra of sets having the Baire property is the smallest σ-algebra containing all open and all meager sets.*

As in the case of measure, there exist non-Borel sets with the Baire property, and using the Axiom of Choice one can show that there exists set that do not have the Baire property.

We conclude this lecture with a note on the relationship between measure and category. From the results so far it seems that they behave quite similarly. This might lead to the conjecture that maybe they more or less coincide. This is not so, in fact, they are quite orthogonal to each other, as the next result shows.

Proposition 7.19: *The real numbers can be partitioned into two subsets, one a Lebesgue nullset and the other one meager.*

Proof. Let (G_n) be a sequence of open sets witnessing that $\mathbb{Q}$ is a nullset, i.e. each G_n is a union of disjoint open intervals that covers $\mathbb{Q}$ and whose total length does not exceed 2^{-n}. Then $G = \bigcap_n G_n$ is a nullset, but at the same time it is an intersection of open dense sets, thus comeager, hence its complement is meager. $\square$

Lecture 8: The Axiom of Choice

In the previous lectures, a number of **regularity principles** for sets of real numbers emerged.

(PS) The *perfect subset* property,

(LM) *Lebesgue measurability*,

(BP) the *Baire* property.

We have seen that the Borel sets in $\mathbb{R}$ have all these properties. In this lecture we will show how to construct counterexamples for each of these principles. The proofs make essential use if the **Axiom of Choice**:

(AC) Every set $\mathcal{X}$ of non-empty sets has a choice-function.

A *choice function* for $\mathcal{X}$ is a function f that assigns every set $Y \in \mathcal{X}$ an element $y \in Y$.

One of the most famous applications of the Axiom of Choice is Vitali's construction of a non-Lebesgue measurable set.

Theorem 8.1 (Vitali): *There exists a set $A \subseteq \mathbb{R}$ that is not Lebesgue measurable.*

Proof. Put

$$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$

It is straightforward to check that this is an equivalence relation on $\mathbb{R}$. Using a choice function on the equivalence classes of $\sim$ intersected with the unit interval $[0,1]$, we pick from each equivalence class a representative from $[0,1]$, and collect them in a set S.

If we let, for $r \in \mathbb{Q}$,

$$S_r = \{s + r : s \in S\},$$

then

$$S_r \cap S_t \quad \text{for } r \neq t.$$

Suppose S is measurable. Then so is each S_r, and $\lambda(S_r) = \lambda(S)$.

If $\lambda(S) = 0$, then $\lambda(\mathbb{R}) = 0$, which is impossible. On the other hand, if $\lambda(S) > 0$, then, by countable additivity,

$$2 = \lambda([0,2]) \geq \lambda\left(\bigcup_{r \in \mathbb{Q}\cap[0,1]} S_r\right) = \sum_{r \in \mathbb{Q}\cap[0,1]} \lambda(S) = \infty,$$

contradiction. $\square$

The Axiom of Choice is equivalent to a number of other principles. We will use the **Well-ordering Principle**:

(WO) Every set X can be well-ordered.

This means that one can define a binary relation $<$ on X so that every non-empty subset of X has a $<$-minimal element.

We use (WO) to construct a set $B \subseteq \mathbb{R}$ such neither B nor $\mathbb{R} \setminus B$ contains a perfect subset. Such sets are called **Bernstein sets**.

Theorem 8.2: *There exists a Bernstein set.*

Proof. Let $\mathcal{P}$ be the set of perfect subsets of $\mathbb{R}$. We can well-order this set, say

$$\mathcal{P} = \{P_\xi : \xi < 2^{\aleph_0}\}.$$

Note that every perfect subset corresponds to Cantor-Scheme, which can be coded by a real number (see Lecture 9). Therefore, there are at most $2^{\aleph_0}$-many perfect subsets of $\mathbb{R}$, and it is not hard to see that there are exactly $2^{\aleph_0}$-many.

Furthermore, we assume each P_ξ is well-ordered.

Pick $a_0 \neq b_0$ from P_0. Assume we have chosen $\xi < 2^{\aleph_0}$, and $\{a_\beta : \beta < \xi\}$ and $\{b_\beta : \beta < \xi\}$ so that

$$a_\beta, b_\beta \in P_\beta \quad \text{and} \quad \text{all } a_\beta, b_\gamma \text{ pairwise distinct},$$

we can choose $a_\xi, b_\xi \in P_\xi$ to be the first two elements of $P_\xi \setminus \bigcup_{\gamma<\xi}\{a_\gamma, b_\gamma\}$. This is possible since a perfect subset of $\mathbb{R}$ has cardinality $2^{\aleph_0}$, and $\xi < 2^{\aleph_0}$.

Put

$$A = \{a_\xi : \xi < 2^{\aleph_0}\} \qquad B = \{b_\xi : \xi < 2^{\aleph_0}\}.$$

Neither A nor B has a perfect subset by construction, and since $A \subseteq \mathbb{R} \setminus B$, B is a Bernstein set. $\square$

Proposition 8.3: *A Bernstein set does not have the Baire property.*

Proof. Assume for a contradiction a Bernstein set B has the Baire property. Then there exists an open set U and a meager F_σ set F such that

$$B \bigtriangleup U \subseteq F.$$

Then $G = U \setminus F$ is G_δ, and $G \subseteq U \cap B \subseteq B$. Furthermore, $M = B \setminus G \subseteq F$ is meager, and thus we have $B = M \cup G$, where M is meager and G is G_δ.

At least one of B, $\mathbb{R} \setminus B$ is not meager. Wlog assume B is not meager. (If not, obtain the representation "meager $\cup\, G_\delta$" above for $\mathbb{R} \setminus B$ and proceed analogously.) Then B contains a non-meager G_δ set G, which must be uncountable. By Theorem 6.2, G is Polish and hence must contain a perfect subset, contradiction. $\square$

The existence of arbitrary choice functions appears to be a rather strong assumption. It has consequences that seem paradoxical in the sense that they conflict with basic intuitions we have about objects and they behavior with respect to size or other characteristics. Arguably the most famous example is the **Banach-Tarski Paradox**, which uses the Axiom of Choice to partition a ball in $\mathbb{R}^3$ into finitely many pieces, and then, using rigid transformations (i.e. rotations and translations), to assemble them into two balls of the original size.

On the other hand, the Axiom of Choice has implies or is even equivalent to many principles that are applied throughout many areas of mathematics, such as the existence of bases of vector spaces, Zorn's Lemma, Tychonoff's Theorem on the compactness of product spaces, the Hahn-Banach Theorem, or the Prime Ideal Theorem.

For some applications, however, a weaker form of the Axiom of Choice is sufficient.

The **Axiom of Countable Choice**:

(AC_ω) Every *countable* family $\mathcal{X}$ of non-empty sets has a choice-function.

Stronger than Countable Choice, but still weaker than the full Axiom of Choice is **Axiom of Dependent Choices**:

(DC) If E is a binary relation on a non-empty set A, and if for every $a \in A$ there exists $b \in A$ such that $a \, E \, b$, then there exists a function $f : \mathbb{N} \to A$ such that for all $n \in \mathbb{N}$, $f(n) \, E \, f(n+1)$.

A seminal result by **(author?)** [Sol70] showed that DC is no longer sufficient to prove the existence of non-regular sets in the above sense. He constructed (though under a large cardinal assumption) a model of $\mathsf{ZF} + \mathsf{DC}$ in which every set of real numbers is Lebesgue measurable, has the Baire property and the perfect subset property.

Lecture 9: Effective Borel sets

Suppose $U \subseteq \mathbb{N}^{\mathbb{N}}$ is open. The there exists a set $W \subseteq \mathbb{N}^{<\mathbb{N}}$ such that

$$U = \bigcup_{\sigma \in W} N_\sigma.$$

Using a standard (effective) coding procedure, we can identify finite sequence of natural numbers with a natural number, and thus can see W as a subset of $\mathbb{N}$.

If we provide a Turing machine with oracle W, we can semi-effectively test for membership in U as follows. Assume we want to determine whether some $\alpha \in \mathbb{N}^{\mathbb{N}}$ is in U. Write α on another oracle tape, and start scanning the W oracle. If we retrieve a σ that coincides with an initial segment of α, we know $\alpha \in U$. On the other hand, if $\alpha \in U$, then we will eventually find some $\alpha\,|_n$ in W. If $\alpha \notin U$, then the search will run forever. In other words, given W, U is *semi-decidable*, or, extending terminology from subsets of $\mathbb{N}$ to subsets of $\mathbb{N}^{\mathbb{N}}$, U is **recusively enumerable** relative to W.

Similarly, we can identify a closed set F with the code for the tree

$$T_F = \{\alpha\,|_n \colon \alpha \in F, n \in \mathbb{N}\}.$$

Then we determining whether $\alpha \in F$ is *co-r.e.* in (the code of) T_F. If $\alpha \notin F$ we will learn so after a finite amount of time.

These simple observations suggest the following general approach to Borel sets.

- Borel sets can be coded by a single infinite sequence in $\mathbb{N}^{\mathbb{N}}$ (or $2^{\mathbb{N}}$).
- Given the code, we can describe the Borel set effectively, by means of oracle computations.
- The connection between degrees of unsolvability and definability results in a close correspondence between arithmetical sets (Σ^0_n) and Borel sets of finite order ($\boldsymbol{\Sigma}^0_n$).

In this lecture we will fully develop this correspondence. Later, we will see that it even extends beyond the finite level.

Borel codes

We fix a computable bijection $\pi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. Furthermore, let $\langle .,. \rangle$ be the standard coding function for pairs,

$$\langle x, y \rangle = \frac{(x+y)(x+y+1)}{2} + y.$$

Borel codes are defined inductively.

Definition 9.1:

(a) A real $\gamma \in \mathbb{N}^{\mathbb{N}}$ is a $\mathbf{\Sigma}^0_1$- **code** if $\gamma(0) = 2$. In this case we say that γ *codes* the open set

$$U_\gamma = \bigcup_{n \in \mathbb{N}} N_{\pi^{-1}(\gamma(n))},$$

or γ is a $\mathbf{\Sigma}^0_1$-*code* for U.

(b) If γ is a $\mathbf{\Sigma}^0_n$-code, then γ' with $\gamma'(0) = 3$, $\gamma'(n+1) = \gamma(n)$ is a $\mathbf{\Pi}^0_n$-code. If γ codes $A \subseteq \mathbb{N}^{\mathbb{N}}$, then we say γ' codes $\neg A$.

(c) If for each $m \geq 0$, γ_m is a $\mathbf{\Pi}^0_n$ code for $A_m \subseteq \mathbb{N}^{\mathbb{N}}$, then γ given by $\gamma(0) = 4$ and

$$\gamma(\langle m, n \rangle + 1) = \gamma_m(n)$$

is a $\mathbf{\Sigma}^0_{n+1}$ code, and it codes the set $\bigcup_m A_m$.

Hence the first position in each code indicates the kind of set it codes – an open set, a complement, or a union.

We also define the *set of Borel codes of finite order*

$$\mathrm{Bc}_\omega = \{\gamma \in \mathbb{N}^{\mathbb{N}} : \gamma \text{ is a } \mathbf{\Sigma}^0_n\text{- or } \mathbf{\Pi}^0_n\text{-code, for some } n \geq 1\}.$$

The following is a straightforward induction.

Proposition 9.2: *A set is* $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$) *if and only if it has a* $\mathbf{\Sigma}^0_n$- ($\mathbf{\Pi}^0_n$) *code.*

Note that the definition actually assigns codes to *representations of sets*. A Borel set can have (and has) multiple codes, just as it has multiple representations. We can, for example, represent an open set by different sets W of initial segments.

Moreover, every $\mathbf{\Sigma}^0_1$ set is also $\mathbf{\Sigma}^0_2$, and thus a set has codes which reflect the "more complicated" definition of the $\mathbf{\Sigma}^0_1$ set as a union of closed sets. It is useful to keep this distinction between a Borel set and its Borel representation in mind.

Each Borel code induces a tree structure that reflects how the corresponding Borel set is built up from open sets. A "4" corresponds to a node with infinitely many nodes immediately below it, a "3" to a node with just one immediate extension, and a "2" represents a terminal node, since the open sets are the "building blocks" of the Borel sets and hence are not split further.

The tree of a Borel code is well-founded (i.e. has no infinite path), since a Borel code is defined via a well-founded recursion. The rank of the tree is a countable ordinal.

How hard is it to decide whether a given real is a Borel code? We will see later that this question is quite difficult. In particular, we will extend the set of Borel codes to transfinite orders and see that the set of all Borel codes is *not* Borel. Deciding whether a tree on $\mathbb{N}$ is well-founded will play a fundamental role in this regard.

Borel sets with computable codes

Suppose γ is a computable, $\mathbf{\Sigma}^0_2$-code for an F_σ set F. Then γ is of the form $(4, \gamma')$, where, for $m \geq 0$, the *m-th column* of γ',

$$\gamma'_m(n) = \gamma'(\langle m, n \rangle)$$

is of the form $(3, \alpha_m)$, each α_m being a $\mathbf{\Sigma}^0_1$-code for an open set. Note that γ' and all α_m are computable, too.

We can formulate membership in F as follows.

$$\alpha \in F \quad \Longleftrightarrow \quad \exists m \, \forall n \, [\gamma'_m(\pi(\alpha \upharpoonright_n)) = 0].$$

Note that the *inner* predicate $R(x, y)$ given by

$$R(x, y) \quad \Longleftrightarrow \quad \gamma'_x(y) = 0$$

is decidable. Hence an F_σ set F with a computable code can be represented in the following form. There exists a recursive predicate $R(x, y)$ such that

$$\alpha \in F \quad \Longleftrightarrow \quad \exists m \, \forall n \, R(m, \alpha \upharpoonright_n).$$

In this formulation we drop the coding function π and identify finite sequences directly with natural numbers, and from now on we will continue to do so. It significantly simplifies notation.

On the other hand, if $R(x,y)$ is a recursive predicate, we can define the set

$$W_m = \{\sigma : R(m,\sigma)\}.$$

Then the set $U_m = \bigcup_{W_m} N_\sigma$ is open, and the set F given by

$$\alpha \in F \quad \Longleftrightarrow \quad \exists n\, \alpha \in \neg U_m \quad \Longleftrightarrow \quad \exists m\, \forall n\, \neg R(m, \alpha|_n)$$

is F_σ.

Thus, there seems to be a close connection between F_σ sets with recursive Borel codes and sets definable by Σ^0_2 formulas over recursive predicates. Given that we introduced the notation $\mathbf{\Sigma}^0_2$ for F_σ sets earlier, this is perhaps not very surprising.

In this analysis, there seems to be nothing specific about the F_σ used in the example. Indeed, it can be extended to Borel sets of finite order, which we will do next. We introduce the **lightface** Borel hierarchy and show that it corresponds to Borel sets of finite order with recursive codes. Using relativization, we then obtain a complete characterization of Borel sets of finite order: *They are precisely those sets definable by arithmetical formulas, relative to a real parameter.*

The effective Borel hierarchy

Definition 9.3: A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is

(a) Σ^0_1 if there exists a recursive predicate $R(x)$ such that

$$\alpha \in A \quad \Longleftrightarrow \quad \exists n\, R(\alpha|_n),$$

(b) Π^0_n if $\neg A$ is Σ^0_n,

(c) Σ^0_{n+1} if there exists a Π^0_n set P such that

$$\alpha \in A \quad \Longleftrightarrow \quad \exists n\, [(n,\alpha) \in P].$$

The following result is at the heart of the effective theory.

Proposition 9.4: *Let $A \subseteq \mathbb{N}^{\mathbb{N}}$. Then*

$$A \text{ is } \Sigma^0_n\ (\Pi^0_n) \quad \Longleftrightarrow \quad A \text{ is } \mathbf{\Sigma}^0_n\ (\mathbf{\Pi}^0_n) \text{ and has a computable } \mathbf{\Sigma}^0_n\text{- } (\mathbf{\Pi}^0_n\text{-}) \text{ code.}$$

Proof. We proceed by induction on the Borel complexity.

Suppose A is Σ^0_1. Let R be recursive such that $A = \{\alpha\colon \exists n\, R(\alpha\,|_n)\}$. If we let $W = \{\sigma : R(\sigma)\}$, then

$$A = \bigcup_{\sigma \in W} N_\sigma,$$

and hence is an open set. Furthermore,

$$\gamma(n) = \begin{cases} 2 & n = 0, \\ 1 & n \geq 1 \,\&\, R(n-1), \\ 0 & n \geq 1 \,\&\, \neg R(n-1), \end{cases}$$

is a computable Borel code for A.

If A is $\boldsymbol{\Sigma}^0_1$ with a computable, $\boldsymbol{\Sigma}^0_1$-code γ, then γ is of the form $(2, \gamma')$, γ' coding a representation of A as a union of basic open cylinders. Then

$$\alpha \in A \quad \Longleftrightarrow \quad \exists n\, [\gamma'(\alpha\,|_n) = 1].$$

Hence we can set $R(\sigma) = \gamma'(\sigma)$.

If A is Π^0_n, then $\neg A$ is Σ^0_n. By induction hypothesis, $\neg A$ has a computable $\boldsymbol{\Sigma}^0_n$-code γ. Then $(3, \gamma)$ is a computable $\boldsymbol{\Pi}^0_n$-code for A.

Conversely, if $\gamma = (3, \gamma')$ is a computable $\boldsymbol{\Pi}^0_n$-code for a $\boldsymbol{\Pi}^0_n$ set A, then γ' is a computable $\boldsymbol{\Sigma}^0_n$-code for the $\boldsymbol{\Sigma}^0_n$ set $\neg A$. By induction hypothesis, $\neg A$ is Σ^0_n and hence A is Π^0_n.

Finally, assume that A is Σ^0_{n+1}. Let P be Π^0_n such that

$$\alpha \in A \quad \Longleftrightarrow \quad \exists n\, [(n, \alpha) \in P].$$

By induction hypothesis, there exists P is $\boldsymbol{\Pi}^0_n$ with a computable $\boldsymbol{\Pi}^0_n$-code $\gamma = (3, 4, \dots)$. Let

$$P_m = \{\beta : (m, \beta) \in P\} = P \cap N_{\langle m \rangle}.$$

Then each P_m is $\boldsymbol{\Pi}^0_n$, since the Borel levels are closed under finite intersections, and we have

$$A = \bigcup_m P_m.$$

Therefore, A is $\boldsymbol{\Sigma}^0_{n+1}$. Furthermore, each P_m has a computable $\boldsymbol{\Pi}^0_n$-code γ_m, which can be computed uniformly in m, and thus $\gamma^* = (4, (\gamma_m(n))_{m,n})$ is a computable, $\boldsymbol{\Sigma}^0_{n+1}$-optimal code for A.

For the converse, let A be $\mathbf{\Sigma}^0_{n+1}$ with a computable $\mathbf{\Sigma}^0_{n+1}$-code $\gamma = (4, \gamma')$. Then each of the columns of γ' is a computable $\mathbf{\Pi}^0_n$-code for a $\mathbf{\Pi}^0_n$ set P_m. Let $P'_m = \{(m, \alpha) \colon \alpha \in P_m\}$. P'_m is $\mathbf{\Pi}^0_n$, too. This can be seen as follows. $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. $\{m\} \times P_m$ is Π^0_n in $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, by replacing each set S in the definition of P_m by $\{m\} \times S$ (note that $\{m\}$ is clopen in $\mathbb{N}$). Borel complexities are preserved under homeomorphic images. (We will discuss the closure properties of Borel sets in detail later.)

A similar argument shows that $P^*_m = \{(k, \alpha) \colon k \neq m \text{ or } (k = m \,\&\, \alpha \in P_m)\}$ is $\mathbf{\Pi}^0_n$ for each n. Now let $P^* = \bigcup_m P^*_m$. Then P^* is $\mathbf{\Pi}^0_n$, and we can effectively and uniformly in m compute an $\mathbf{\Pi}^0_n$-code for it. By induction hypothesis, P^* is Π^0_n, and we have

$$\alpha \in A \quad \Leftrightarrow \quad \exists m \, (m, \alpha) \in P^*,$$

as desired. □

Relativization

Using relativized computations via oracles, we can define a relativized version of the effective Borel hierarchy. This way we can capture *all* Borel sets of finite order, not just the ones with computable codes.

Definition 9.5: Let $\gamma \in \mathbb{N}^{\mathbb{N}}$. A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is

(a) $\Sigma^0_1(\gamma)$ if there exists a predicate $R(x)$ *recursive* in γ such that

$$\alpha \in A \quad \Leftrightarrow \quad \exists n \, R(\alpha \,|_n),$$

(b) $\Pi^0_n(\gamma)$ if $\neg A$ is $\Sigma^0_n(\gamma)$,

(c) $\Sigma^0_{n+1}(\gamma)$ if there exists a $\Pi^0_n(\gamma)$ set P such that

$$\alpha \in A \quad \Leftrightarrow \quad \exists n \, [(n, \alpha) \in P].$$

A straightforward relativization gives the following analogue of Proposition 9.4.

Proposition 9.6: *Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ and $\gamma \in \mathbb{N}^{\mathbb{N}}$. Then*

$$A \text{ is } \Sigma^0_n(\gamma)\ (\Pi^0_n(\gamma)) \quad \Leftrightarrow \quad A \text{ is } \mathbf{\Sigma}^0_n\ (\mathbf{\Pi}^0_n) \text{ and has a } \mathbf{\Sigma}^0_n\text{- } (\mathbf{\Pi}^0_n\text{-}) \text{ code recursive in } \gamma.$$

We can now present the *fundamental theorem of effective descriptive set theory*.

Theorem 9.7: *A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\boldsymbol{\Sigma}^0_n$ ($\boldsymbol{\Pi}^0_n$) if and only if it is $\Sigma^0_n(\gamma)$ ($\Pi^0_n(\gamma)$) for some $\gamma \in \mathbb{N}^{\mathbb{N}}$.*

Proof. If A is $\boldsymbol{\Sigma}^0_n$, then by Proposition 9.2 it has a $\boldsymbol{\Sigma}^0_n$-code γ, and by Proposition 9.6 A is $\Sigma^0_n(\gamma)$. The other direction follows immediately from Proposition 9.6.

The argument for $\boldsymbol{\Pi}^0_n$ is completely analogous. □

Definability in Arithmetic

One of the fundamental insights of recursion theory is the close relation between computability and definability in arithmetic. The recursively enumerable subsets of $\mathbb{N}$ are precisely the sets Σ_1-definable over the standard model of arithmetic, $(\mathbb{N}, +, \cdot, 0, 1)$, and **Post's Theorem** uses this result to establish a rigid connection between levels of arithmetical complexity and computational complexity.

As indicated above, we can use this relation to give a characterization of the Borel sets of finite order in terms of definability. Since we are dealing with subsets of $\mathbb{N}^{\mathbb{N}}$, that is, with *sets* of functions on $\mathbb{N}$ rather than just functions on $\mathbb{N}$, we will work in the framework of *second order arithmetic*.

The **language of second order arithmetic**has two kinds of variables: *number variables* $x, y, z, \ldots$ (and sometimes k, l, m, n if they are not used as metavariables), to be interpreted as elements of $\mathbb{N}$, and *function variables* $\alpha, \beta, \gamma, \ldots$, intended to range over functions from $\mathbb{N}$ into $\mathbb{N}$, i.e. elements of Baire space, i.e. reals. The non-logical symbols are the binary function symbols $+, \cdot$, the binary relation symbol $<$, the *application function* symbol ap, and the constants $\underline{0}, \underline{1}$. *Numerical terms* are defined in usual way using $+, \cdot, \underline{0}, \underline{1}$, and involve only number variables. *Atomic formulas* are $t_1 = t_2$, $t_1 < t_2$, and $\mathrm{ap}(\alpha, t_1) = t_2$, where t_1, t_2 are numerical terms.

The **standard model of second order arithmetic** is the structure

$$\mathcal{A}^2 = (\mathbb{N}, \mathbb{N}^{\mathbb{N}}, \mathrm{ap}, +, \cdot, <, 0, 1),$$

where $+$ and $\cdot$ are the usual operations on natural numbers, $<$ is the standard ordering of $\mathbb{N}$. The two domains are connected by the binary operation $\mathrm{ap} : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$, defined as

$$\mathrm{ap}(\alpha, x) = \alpha(x).$$

A relation $R \subseteq \mathbb{N}^m \times (\mathbb{N}^{\mathbb{N}})^n$ is *definable over* $\mathcal{A}^2$ if there exists a formula φ of second order arithmetic such that for any $x_1, \ldots, x_m \in \mathbb{N}$ and $\alpha_1, \ldots \alpha_n \in \mathbb{N}^{\mathbb{N}}$,

$$R(x_1, \ldots, x_m, \alpha_1, \ldots \alpha_n) \quad \text{iff} \quad \mathcal{A}^2 \models \varphi[x_1, \ldots, x_m, \alpha_1, \ldots \alpha_n].$$

Theorem 9.8: *A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ^0_n (Π^0_n) if and only if it is definable over $\mathcal{A}^2$ by a Σ^0_n (Π^0_n) formula.*

Here, Σ^0_n (Π^0_n) formula means that we can *only quantify over number variables*, as opposed to Σ^1_n (Π^1_n) formulas, where we can also quantify over function variables.

The proof is a straightforward extension of the standard argument for subsets of $\mathbb{N}$.

To formulate the fundamental Theorem 9.7 in terms of definability, we need the concept of **relative definability**. We add a new constant function symbol $\underline{\gamma}$ to the language. Given a function γ, a relation is **definable in** γ if it is definable over the structure

$$\mathcal{A}^2(\gamma) = (\mathbb{N}, \mathbb{N}^{\mathbb{N}}, \mathrm{ap}, +, \cdot, <, 0, 1, \underline{\gamma}),$$

where the symbol $\underline{\gamma}$ is interpreted as γ.

Then the following holds.

Theorem 9.9: *A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\boldsymbol{\Sigma}^0_n$ ($\boldsymbol{\Pi}^0_n$) if and only if it is definable in γ by a Σ^0_n (Π^0_n) formula, for some $\gamma \in \mathbb{N}^{\mathbb{N}}$.*

This theorem facilitates the description of Borel sets considerably. As an example, consider the set

$$A = \{\alpha \colon \alpha \text{ eventually constant}\}.$$

We have

$$\alpha \in A \quad \Leftrightarrow \quad \exists n \forall m [m \geq n \Rightarrow \alpha(n) = \alpha(m)]$$

The right hand side is a Σ^0_2-formula. Hence the set A is $\boldsymbol{\Sigma}^0_2$ (even Σ^0_2).

Lecture 10: The Structure of Borel Sets

In this lecture we further investigate the structure of Borel sets. We will use the results of the previous lecture to derive various closure properties and other structural results. As an application, we see that the Borel hierarchy is indeed proper.

Notation

Before we go on, we have to address some notational issues. So far we have used notation quite liberally, especially when it came to product sets. We will continue to do so, but we want to put this on a firmer footing.

Using coding, we can identify any product space $\mathbb{N}^m \times (\mathbb{N}^\mathbb{N})^n$ with $\mathbb{N}^\mathbb{N}$. One way to do this is to fix, for each $n \geq 1$, an effective homeomorphism $\theta_n : (\mathbb{N}^\mathbb{N})^n \to \mathbb{N}^\mathbb{N}$ and map

$$(k_1, \dots, k_m, \alpha_1, \dots, \alpha_n) \mapsto (k_1, \dots, k_m, \theta_n(\alpha_1, \dots, \alpha_n)).$$

Here $(k_1, \dots, k_m, \theta_n(\alpha_1, \dots, \alpha_n))$ is just a suggestive way of writing the concatenation

$$\langle k_1 \rangle^\frown \cdots ^\frown \langle k_m \rangle^\frown \theta_n(\alpha_1, \dots, \alpha_n).$$

We have already used this notation in the previous lecture. In the following, we will continue to switch freely between product sets and their coded counterparts, as subsets of $\mathbb{N}^\mathbb{N}$.

Another notation identifies sets and relations. We will identify sets $A \subseteq \mathbb{N}^m \times (\mathbb{N}^\mathbb{N})^n$ with the relation they induce and write $A(k_1, \dots, k_m, \alpha_1, \dots, \alpha_n)$ instead of $(k_1, \dots, k_m, \alpha_1, \dots, \alpha_n) \in A$. Conversely, we will identify relations with the set they induce.

Normal forms

Theorem 9.9 tells us that a set $A \subseteq \mathbb{N}^\mathbb{N}$ is $\mathbf{\Sigma}^0_n$ if and only if it is definable by a Σ^0_n formulas over $\mathcal{A}^2$, relative to some parameter. That means that there exists a *bounded formula* $\varphi(x_1, \dots, x_n, \alpha, \underline{\gamma})$ (i.e. all quantifiers are bounded) such that

$$A(\alpha) \quad \Leftrightarrow \quad \exists x_1 \dots \mathsf{Q} x_n \, \varphi(x_1, \dots, x_n, \alpha, \gamma) \text{ holds (in the standard model).}$$

Here γ is the parameter, and Q is "$\exists$" if n is odd, and "$\forall$" if n is even.

Similarly, $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Pi}^0_n$ if and only if it is definable as

$$A(\alpha) \quad \Longleftrightarrow \quad \forall x_1 \ldots \mathsf{Q} x_n \, \varphi(x_1, \ldots, x_n, \alpha, \gamma) \text{ holds (in the standard model).}$$

where $\varphi(x_1, \ldots, x_n, \alpha, \underline{\gamma})$ is bounded, and Q is "$\forall$" if n is odd, and "$\exists$" if n is even.

What do sets defined by bounded formulas look like? An atomic formula (without parameters) either contains no function variable at all, or it is of the form $\alpha(t_1) = t_2$. This implies that the truth of an atomic formula is determined by *finitely many positions* in α. This remains true if we consider logical combinations of atomic formulas, or even bounded quantification. Hence *a bounded formula defines an open subset* of $\mathbb{N}^{\mathbb{N}}$. On the other hand, the reals for which a bounded formula does not hold are definable by a bounded formula, too, since the negation of a bounded formula is again a bounded formula. We conclude that **bounded formulas define clopen subsets of $\mathbb{N}^{\mathbb{N}}$.** On the other hand, if we have $\mathbf{\Sigma}^0_1$-code for a set A and its complement, we can decide the relation $A(\alpha\restriction_n)$ recursively in the code.

Hence we can formulate the Normal Form above as follows. $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Sigma}^0_n$ if and only if there exists a clopen set $R \subseteq \mathbb{N}^n \times \mathbb{N}^{\mathbb{N}}$

$$A(\alpha) \quad \Longleftrightarrow \quad \exists x_1 \ldots \mathsf{Q} x_n \, R(x_1, \ldots, x_n, \alpha),$$

and similarly for $\mathbf{\Pi}^0_n$ sets.

Closure properties

We can use the Normal Form to derive several closure properties of $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$).

If $P \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, we define the **projection of P along** $\mathbb{N}$ $\exists^{\mathbb{N}} P$ as

$$\exists^{\mathbb{N}} P = \{\alpha : \exists n \, P(n, \alpha)\}.$$

The dual operation is

$$\forall^{\mathbb{N}} P = \{\alpha : \forall n \, P(n, \alpha)\}.$$

Proposition 10.1: *For each $n \geq 1$, $\mathbf{\Sigma}^0_n$ is closed under $\exists^{\mathbb{N}}$, and $\mathbf{\Pi}^0_n$ is closed under $\forall^{\mathbb{N}}$.*

Proof. We prove the result for Σ^0_n (lightface). The boldface case follows by relativization, and the proof for $\mathbf{\Pi}^0_n$ is completely dual.

Let $\varphi(x_1, \dots, x_n, z, \alpha)$ be a bounded formula such that

$$A(z, \alpha) \quad \Leftrightarrow \quad \exists x_1 \dots \mathsf{Q} x_n \; \varphi(x_1, \dots, x_n, z, \alpha) \text{ holds.}$$

Then

$$\exists^{\mathbb{N}} A(\alpha) \quad \Leftrightarrow \quad \exists x_0 \exists x_1 \dots \mathsf{Q} x_n \; \varphi(x_1, \dots, x_n, x_0, \alpha)$$

We can collect two existential number quantifiers into one by using the pairing function $\langle ., . \rangle$, or rather, its inverses, which we will denote by $(.)_0$ and $(.)_1$. (Recall that the pairing function is definable by a bounded formula.) Then

$$\exists^{\mathbb{N}} A(\alpha) \quad \Leftrightarrow \quad \exists z_1 \dots \mathsf{Q} z_n \; \varphi((z_1)_1, \dots, z_n, (z_1)_0, \alpha),$$

as desired. □

One can use similar applications of coding and quantifier manipulation to prove a number of other closure properties, Often they follow also directly from the topological definitions, but it is good to have several techniques at hand.

Proposition 10.2:

(a) *For all $n \geq 1$, $\mathbf{\Sigma}^0_n$ is closed under countable unions and finite intersections.*

(b) *For all $n \geq 1$, $\mathbf{\Pi}^0_n$ is closed under finite unions and countable intersections.*

(c) *For all $n \geq 1$, $\mathbf{\Delta}^0_n$ is closed under finite unions, finite intersections, and complements.*

Proof. One can prove this by induction along the hierarchy. To obtain the closure under finite unions and intersections, one can use the following logical equivalences.

$$\begin{aligned} \exists x\, P(x) \wedge \exists y\, R(y) \quad &\Leftrightarrow \quad \exists x \exists y\, (P(x) \wedge R(y)) \\ \forall x\, P(x) \vee \forall y\, R(y) \quad &\Leftrightarrow \quad \forall x \forall y\, (P(x) \vee R(y)) \end{aligned}$$

□

Given $P \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, the **bounded projection** along $\mathbb{N}$ is defined as

$$\exists^{\leq} P = \{(n, \alpha) \colon \exists m \leq n\, P(m, \alpha)\}.$$

and the dual is

$$\forall^{\leq} P = \{(n, \alpha) \colon \forall m \leq n\, P(m, \alpha)\}.$$

Proposition 10.3: *For all $n \geq 1$, $\mathbf{\Sigma}^0_n$, $\mathbf{\Pi}^0_n$, and $\mathbf{\Delta}^0_n$ are closed under $\exists^{\leq}$ and $\forall^{\leq}$.*

Proof. In this case we use the coding function $\pi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. We can define a *partial inverse*

$$(k)_i = \begin{cases} \sigma_i & \text{if } k = \pi(\sigma) \text{ for a finite sequence } \sigma = (\sigma_0, \dots, \sigma_{n-1}) \\ & \quad \text{and } i < n \\ 0 & \text{otherwise.} \end{cases}$$

Using this decoding, we have the following equivalence, which immediately imply the closure properties for $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$, respectively, and hence also for $\mathbf{\Delta}^0_n$.

$$\begin{aligned} \forall m \leq n \, \exists k \, P(m,k) \quad &\Leftrightarrow \quad \exists k \, \forall m \leq n \, P(m,(k)_m) \\ \exists m \leq n \, \forall k \, P(m,k) \quad &\Leftrightarrow \quad \forall k \, \exists m \leq n \, P(m,(k)_m) \end{aligned}$$

□

Finally, the levels of the Borel hierarchy are closed under continuous preimages.

Proposition 10.4: *For all $n \geq 1$, for any $A \subseteq \mathbb{N}^{\mathbb{N}}$, and for any continuous $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, if A is $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$, $\mathbf{\Delta}^0_n$) then $f^{-1}(A)$ is $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$, $\mathbf{\Delta}^0_n$).*

Proof. This follows easily by induction on n, since open and closed sets are closed under continuous preimages.

However, we can also argue via definability, since by Proposition 4.7 one can represent a continuous function through a monotone mapping ψ from finite strings to finite strings. We have

$$f^{-1}(A) = \{\alpha : A(f(\alpha))\}.$$

Let R be clopen such that

$$A(\alpha) \quad \Leftrightarrow \quad \exists x_1 \dots \mathsf{Q} x_n \, R(x_1, \dots, x_n, \alpha).$$

Since clopen predicates depend only on a finite initial segment of α, we can substitute $f(\alpha)$ for α. The resulting formula defines $f^{-1}(A)$, and is equivalent to a Σ^0_n-formula relative to a parameter coding the mapping ψ. □

Universal sets

Let Γ be a family of subsets defined in various Polish spaces. Of course we have in mind the classes $\mathbf{\Sigma}^0_n$ or $\mathbf{\Pi}^0_n$, but the concept of a *universal set* can be defined quite generally.

Definition 10.5: Let Y be a set. A set $U \subseteq X \times Y$ is Y**-universal for** Γ if $U \in \Gamma$, and for every set A in Γ, there exists a $y \in Y$ such that

$$A = \{x : (x, y) \in U\}.$$

A universal set for Γ can be thought of as a **parametrization** of Γ, the second component providing a *code* or *parameter* for each set in Γ.

A well-known example of a universal set is the *generalized halting problem*,

$$K_0 = \{(x, e) : \text{ the } e\text{-th Turing machine halts on input } x\}.$$

In the sense of the above definition, K_0 is $\mathbb{N}$-universal for the family of recursively enumerable sets.

We have seen in the previous lecture that there is a strong connection between r.e. sets and Σ^0_1 sets. The relation is based on the fact that each Σ^0_1 set in $\mathbb{N}^{\mathbb{N}}$ has a *code* that is r.e. We can use the code to obtain a universal set for $\mathbf{\Sigma}^0_1$.

Proposition 10.6: *For any $n \geq 1$, there exists a set $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ that is $\mathbb{N}^{\mathbb{N}}$-universal for $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$).*

Proof. We can use the Borel codes defined in the previous lecture.

First of all, notice that for each $n \geq 1$, the set of all $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$)-codes is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. This follows easily from the definition of the Borel codes. Hence, if we fix n, every $\gamma \in \mathbb{N}$ represents a $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$)-code of a $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$) set, and every such set in turn has a code $\gamma \in \mathbb{N}$.

Now we let, for fixed n,

$$U = \{(\alpha, \gamma) : \ \alpha \text{ is in the } \mathbf{\Sigma}^0_n \ (\mathbf{\Pi}^0_n) \text{ set coded by } \gamma\}.$$

It follows easily from Theorem 9.9 that U is $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$), too, and it is clear from the definition of U that it parametrizes $\mathbf{\Sigma}^0_n$ ($\mathbf{\Pi}^0_n$). □

The result can be generalized to hold for arbitrary Polish spaces X, i.e. for any $n \geq 1$, there exists a set $U \subseteq \mathbb{N}^{\mathbb{N}} \times X$ that is $\mathbb{N}^{\mathbb{N}}$-universal for $\mathbf{\Sigma}^0_n(X)$ ($\mathbf{\Pi}^0_n(X)$). To achieve this, one has to define Borel codes for X. This can be done by fixing a countable basis (V_n) of the topology of X, and assign a sequence $\gamma \in \mathbb{N}^{\mathbb{N}}$ the open set

$$U_\gamma = \bigcup_{n \in \mathbb{N}} V_{\gamma(n)}.$$

The definition of codes for higher levels is then similar to Definition 9.1.

As in the case of the halting problem, we can use the existence of universal sets to show that the levels of the Borel hierarchy are proper. The crucial point is that we can use universal sets to *diagonalize*.

Theorem 10.7: *For any $n \geq 1$, $\mathbf{\Sigma}^0_n \neq \mathbf{\Pi}^0_n$.*

Proof. Let U be an $\mathbb{N}^{\mathbb{N}}$-universal set for $\mathbf{\Sigma}^0_n$. Put

$$D = \{\alpha \colon (\alpha, \alpha) \in U\}.$$

Since U is $\mathbf{\Sigma}^0_n$, D is $\mathbf{\Sigma}^0_n$, too. Then $\neg D$ is $\mathbf{\Pi}^0_n$, but cannot be $\mathbf{\Sigma}^0_n$, for then there would exist β such that

$$\neg D = \{\alpha \colon (\alpha, \beta) \in U\},$$

and thus

$$\beta \in D \quad \Longleftrightarrow \quad (\beta, \beta) \in U \quad \Longleftrightarrow \quad \beta \in \neg D,$$

a contradiction. □

The diagonal set D can obviously be defined for any universal set U, and hence the same proof yields a $\mathbf{\Pi}^0_n$ set that is not $\mathbf{\Sigma}^0_n$.

Corollary 10.8: *For any $n \geq 1$,*

$$\mathbf{\Delta}^0_n \subsetneq \mathbf{\Sigma}^0_n \subsetneq \mathbf{\Delta}^0_{n+1}$$
$$\mathbf{\Delta}^0_n \subsetneq \mathbf{\Pi}^0_n \subsetneq \mathbf{\Delta}^0_{n+1}.$$

Proof. Since $\mathbf{\Sigma}^0_n \nsubseteq \mathbf{\Pi}^0_n$ and $\mathbf{\Pi}^0_n \nsubseteq \mathbf{\Sigma}^0_n$, $\mathbf{\Delta}^0_n \subsetneq \mathbf{\Sigma}^0_n, \mathbf{\Pi}^0_n$. On the other hand if $\mathbf{\Sigma}^0_n = \mathbf{\Delta}^0_{n+1}$, then $\mathbf{\Sigma}^0_n$ would be closed under complements, and hence $\mathbf{\Sigma}^0_n = \mathbf{\Pi}^0_n$, contradicting Theorem 10.7. □

Borel sets of transfinite order

We saw that the *Borel sets of finite order*

$$\mathrm{Borel}_\omega = \bigcup_{n<\omega} \mathbf{\Sigma}^0_n$$

form a proper hierarchy. This fact also implies that Borel_ω does not exhaust all Borel sets.

Proposition 10.9: *There exists a Borel set B that is not $\mathbf{\Sigma}^0_n$ for any $n \in \mathbb{N}$.*

Proof. For every $n \in \mathbb{N}$, pick a set B_n in $\mathbf{\Pi}^0_n \setminus \mathbf{\Sigma}^0_n$. Put

$$B = \bigcup_{n \in \mathbb{N}} \{(n, \alpha) \colon \alpha \in B_n\}.$$

Each of the sets in the union is Borel and hence B is Borel. If B were of finite order, it would be $\mathbf{\Sigma}^0_k$ for some $k \geq 1$. Since each $\mathbf{\Sigma}^0_n$ is closed under finite intersections, it follows that for all $m \geq 1$,

$$B \cap N_{\langle m \rangle}$$

is $\mathbf{\Sigma}^0_k$. But $B \cap N_{\langle m \rangle}$ is homeomorphic to B_m, hence B_m in $\mathbf{\Sigma}^0_k$ for all $m \geq 1$, contradiction. □

We can extend the Borel hierarchy to arbitrary ordinals.

Definition 10.10: Let X be a Polish space. Given an ordinal ξ, we define

$$\begin{aligned}
\mathbf{\Sigma}^0_\xi(X) &= \{\bigcup_k A_k \colon A_k \in \mathbf{\Pi}^0_{\zeta_k}(X),\ \zeta_k < \xi\},\\
\mathbf{\Pi}^0_\xi(X) &= \{\neg A \colon A \in \mathbf{\Sigma}^0_\xi(X)\} = \neg\, \mathbf{\Sigma}^0_n(X),\\
\mathbf{\Delta}^0_\xi(X) &= \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi.
\end{aligned}$$

It actually suffices to consider ordinals up to ω_1, the first uncountable ordinal.

Proposition 10.11: *For every Borel set B there exists $\xi < \omega_1$ such that $B \in \mathbf{\Sigma}^0_\xi$.*

Proof. If B is open, this is clear. It is also clear if B is the complement of a Borel for which the statement has been verified.

Assume finally that

$$B = \bigcup_n B_n, \quad \text{where each } B_n \text{ is Borel,}$$

and assume the statement holds for each B_n. For each n, let ξ_n be a countable ordinal such that

$$B_n \in \mathbf{\Pi}^0_{\xi_n}.$$

Then

$$B \in \mathbf{\Sigma}^0_\xi, \quad \text{where } \xi = \sup\{\xi_n + 1 : n \in \mathbb{N}\}.$$

Since each ξ_n is countable, ξ is countable. □

Borel sets of infinite order have the same closure properties as their counterparts of finite order. The proofs, however have to proceed by induction using the topological properties of $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$, since the characterization via definability in arithmetic is no longer available – the arithmetical hierarchy reaches only to ω.

Similarly, the Hierarchy Theorem 10.7 extends to the transfinite levels. For the finite levels, this followed from the existence of universal sets for each level.

Proposition 10.12: *For each $\xi < \omega_1$, there exists a $\mathbb{N}^\mathbb{N}$-universal set for $\mathbf{\Sigma}^0_\xi$ ($\mathbf{\Pi}^0_\xi$).*

Proof. If U is $\mathbb{N}^\mathbb{N}$-universal for $\mathbf{\Sigma}^0_\xi$, then

$$\neg U = \{(\alpha, \gamma) : (\alpha, \gamma) \notin U\}$$

is $\mathbb{N}^\mathbb{N}$-universal for $\mathbf{\Pi}^0_\xi$, since for any $\mathbf{\Pi}^0_\xi$ set A, $B = \neg A$ is $\mathbf{\Sigma}^0_\xi$ and hence there exists a γ such that

$$B = \{\beta : (\beta, \gamma) \in U\}$$

and hence

$$A = \{\alpha : (\alpha, \gamma) \notin U\}.$$

It remains to show that each $\mathbf{\Sigma}^0_\xi$ has an $\mathbb{N}^\mathbb{N}$-universal set. By induction hypothesis, for every $\eta < \xi$ exists a $\mathbb{N}^\mathbb{N}$-universal set U_η for $\mathbf{\Pi}^0_\eta$. Since ξ is countable, we can pick a monotone sequence of ordinals (ξ_n) such that $\xi = \sup\{\xi_n + 1 : n < \omega\}$. Define

$$U_\xi = \{(\alpha, \gamma) : \exists n (\alpha, (\gamma)_n) \in U_{\xi_n}\},$$

where $(\gamma)_n$ denotes the nth column of γ. It is straightforward to check that U_ξ is $\mathbb{N}^\mathbb{N}$-universal for $\mathbf{\Sigma}^0_\xi$. (Note that any set A in $\mathbf{\Sigma}^0_\xi$ can be represented as $\bigcup_n A_n$ with $A_n \in \mathbf{\Pi}^0_{\xi_n}$, since $(\xi_n + 1)$ is cofinal in ξ.) □

This general proof of existence of universal sets does not use Borel codes, since those were defined only for Borel sets of finite order. The proof of Proposition 10.12 provides an idea how we could extend the definition of a code to transfinite orders: Take unions of codes along a cofinal sequence. However, we would like to this in an effective way, and it is not clear how to do this for infinite ordinals in general.

We will later return to this question, when we introduce *computable ordinals*.

Lecture 11: Continuous Images of Borel Sets

In 1916, Nikolai Lusin asked his student Mikhail Souslin to study a paper by Henri Lebesgue. Souslin found a number of errors, including a lemma that asserted that the projection of a Borel is again Borel. In this lecture we will study the behavior of Borel sets under continuous functions. We will see that on the one hand every Borel set is the continuous image of a closed set, but that on the other hand continuous images of Borel sets are not always Borel.

This gives rise to a new family of sets, the *analytic* sets, which form a proper superclass of the Borel sets with interesting properties.

Borel sets as continuous images of closed sets

We have seen in Theorem 2.6 that every Polish space is the continuous image of Baire space $\mathbb{N}^{\mathbb{N}}$. As we will see now, we can strengthen this result.

Theorem 11.1: *Let X be a Polish space. Then there exists a closed subset $F \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous bijection $f : F \to X$ that can be extended to a continuous surjection $g : \mathbb{N}^{\mathbb{N}} \to X$.*

Proof. We have seen (Theorem 2.4) that every uncountable Polish space contains a homeomorphic embedding of Cantor space. This was achieved by means of a *Cantor scheme*. We take up this idea again and adapt it to the Baire space.

A **Lusin scheme** on a set X is a family $(F_\sigma)_{\sigma \in \mathbb{N}^{<\mathbb{N}}}$ of subsets of X such that

(i) $\sigma \subseteq \tau$ implies $F_\sigma \supseteq F_\tau$,

(ii) For all $\tau \in \mathbb{N}^{<\mathbb{N}}$, $i \neq j \in \mathbb{N}$, $F_{\tau^\frown \langle i \rangle} \cap F_{\tau^\frown \langle j \rangle} = \emptyset$.

If it has the additional property that

(iii) $\operatorname{diam}(F_{\alpha|_n}) \to 0$ for $n \to \infty$,

then we can, similarly to a Cantor scheme, define the set

$$D = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \bigcap_{n \in \mathbb{N}} F_{\alpha|_n} \neq \emptyset\}$$

and an **associated map** $f : D \to X$ by

$$\{f(\alpha)\} = \bigcap_{n \in \mathbb{N}} F_{\alpha|_n}.$$

Properties (i)-(iii) ensure that f is continuous and injective.

To prove the theorem we devise a Lusin scheme on X such that D will be closed, and f will be a surjection, too. This is ensured by the following additional properties.

(a) $F_\emptyset = X$,

(b) Each F_τ ist $\mathbf{\Sigma}^0_2$,

(c) For each τ, $\mathrm{diam}(F_\sigma) \leq 1/2^{|\sigma|}$,

(d) $F_\tau = \bigcup_{i\in\mathbb{N}} F_{\tau^\frown\langle i\rangle} = \bigcup_{i\in\mathbb{N}} \overline{F_{\tau^\frown\langle i\rangle}}$.

For this we have to show that every $\mathbf{\Sigma}^0_2$ set $F \subseteq X$ can be written, for given $\varepsilon > 0$, as $F = \bigcup_{i\in\mathbb{N}} F_i$, where the F_i are pairwise disjoint $\mathbf{\Sigma}^0_2$ sets of diameter $< \varepsilon$ so that $\overline{F_i} \subseteq F$:

Let $F = \bigcup_{i\in\mathbb{N}} C_i$, where C_i is closed, and $C_i \subseteq C_{i+1}$. Then $F = \bigcup_{i\in\mathbb{N}}(C_{i+1}\setminus C_i)$. Let (U_n) be a covering of X with open sets of diameter $< \varepsilon$. Put $D^{(i)}_n = U_n \cap (C_{i+1}\setminus C_i)$. Then $D^{(i)}_n$ is $\mathbf{\Delta}^0_2$. Now let $E^{(i)}_n = D^{(i)}_n \setminus (D^{(i)}_1 \cup \cdots \cup D^{(i)}_{n-1})$. Then $C_{i+1}\setminus C_i = \bigcup_{n\in\mathbb{N}} E^{(i)}_n$ where the $E^{(i)}_j$ are $\mathbf{\Sigma}^0_2$ sets of diameter $< \varepsilon$. Therefore,

$$F = \bigcup_{i,n\in\mathbb{N}} E^{(i)}_n \quad\text{and}\quad \overline{E^{(i)}_n} \subseteq \overline{C_{i+1}\setminus C_i} \subseteq C_{i+1} \subseteq F.$$

The mapping f associated with this Lusin scheme is surjective due to (a) and (d). Furthermore, the domain D of f is closed: Suppose $\alpha_n \in D$, $\alpha_n \to \alpha$. Then $f(\alpha_n)$ is Cauchy, since for $\varepsilon > 0$, there exists N with $\mathrm{diam}(F_{\alpha|_N}) < \varepsilon$ and n_0 such that $\alpha_n\,|_N = \alpha\,|_N$ for all $n \geq n_0$, so that $d(f(\alpha_n), f(\alpha_m)) < \varepsilon$ whenever $n, m \geq n_0$. Since X is Polish $f(\alpha_n) \to y$ for some $y \in X$.

By (d) we have $y \in \bigcap_n \overline{F_{\alpha|_n}} = \bigcap_n F_{\alpha|_n}$, hence $\alpha \in D$ and $f(\alpha) = y$.

It remains to show that we can extend f to a continuous surjection $g : \mathbb{N}^\mathbb{N} \to X$. Say a closed subset C of a topological space Y is a **retract** of Y if there exists a continuous surjection $g : Y \to F$ such that $g\,|_C = \mathrm{id}$.

Lemma 11.2: *Every non-empty closed subset of* $\mathbb{N}^\mathbb{N}$ *is a retract of* $\mathbb{N}^\mathbb{N}$.

If we combine the retract function with f, we then obtain the desired surjection $\mathbb{N}^\mathbb{N} \to X$.

Proof of Lemma. Let $C \subseteq \mathbb{N}^{\mathbb{N}}$ be closed, and let T be a pruned tree such that $[T] = C$. We define a monotone mapping $\varphi : \mathbb{N}^{<\mathbb{N}} \to T$ such that $\varphi(\sigma) = \sigma$ for all $\sigma \in T$. Then the induced (continuous) mapping $\varphi^* : \mathbb{N}^{\mathbb{N}} \to C$ is the desired retract.

Define φ by induction. Let $\varphi(\langle\varnothing\rangle) = \langle\varnothing\rangle$. Given $\varphi(\tau)$, let

$$\varphi(\tau^\frown\langle m\rangle) = \begin{cases} \tau^\frown\langle m\rangle & \text{if } \tau^\frown\langle m\rangle \in T, \\ \text{any } \varphi(\tau)^\frown\langle k\rangle \in T & \text{otherwise.} \end{cases}$$

Note that k must exist since T is pruned. □

Refining the topology as in Lecture 6, we can extend the result from Polish spaces to Borel sets.

Corollary 11.3 (Lusin and Souslin)**:** *For every Borel subset B of a Polish space X there exists a closed set $F \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous bijection $f : F \to B$. Furthermore, f can be extended to a continuous surjection $g : \mathbb{N}^{\mathbb{N}} \to B$.*

Proof. Enlarge the topology $\mathcal{O}$ of X to a topology $\mathcal{O}_B$ for which B is clopen. By Theorem 6.2, $(B, \mathcal{O}_B|_B)$ is a Polish space. By the previous theorem, there exists a closed set $F \subset \mathbb{N}^{\mathbb{N}}$ and a continuous bijection $f : \mathbb{N}^{\mathbb{N}} \to (B, \mathcal{O}_B|_B)$. Since $\mathcal{O} \subseteq \mathcal{O}_B$, $f : F \to B$ is continuous for $\mathcal{O}$, too. □

This theorem can be reversed in the following sense.

Theorem 11.4 (Lusin and Suslin)**:** *Suppose X, Y are Polish and $f : X \to Y$ is continuous. If $A \subseteq X$ is Borel and $f|_A$ is injective, then $f(A)$ is Borel.*

Images of Borel sets under arbitrary continuous functions

As announced in the introduction, Borel sets are *not* closed under *arbitrary* continuous mappings.

Theorem 11.5 (Souslin)**:** *The Borel sets are not closed under continuous images.*

Proof. Let $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be $\mathbb{N}^{\mathbb{N}}$-universal for $\mathbf{\Pi}^0_1(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}})$. Define

$$F := \{(\alpha, \beta) : \exists \gamma\, (\alpha, \gamma, \beta) \in U\}.$$

We claim that this set is $\mathbb{N}^{\mathbb{N}}$*-universal for the set of all continuous images of closed subsets of* $\mathbb{N}^{\mathbb{N}}$: On the one hand F is a projection of a closed set, and projections are continuous. This also also implies that all the sets $F_\beta = \{\alpha\colon (\alpha,\beta) \in F\}$ are continuous images of a closed set. On the other hand, if $f : C \to \mathbb{N}^\mathbb{N}$ is continuous with $C \subseteq \mathbb{N}^\mathbb{N}$ closed (possibly empty) and $f(C) = A$, then

$$\alpha \in A \quad\Leftrightarrow\quad \exists\gamma\,(\gamma,\alpha) \in \mathrm{Graph}(f) \quad\Leftrightarrow\quad \exists\gamma\,(\alpha,\gamma) \in \mathrm{Graph}(f^{-1}).$$

Since f is continuous, Graph(f) and hence also Graph(f^{-1}) are closed subsets of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$. Thus, by the universality of U, there exists β such that

$$\mathrm{Graph}(f^{-1}) = U_\beta = \{(\alpha,\gamma)\colon (\alpha,\gamma,\beta) \in U\},$$

and hence

$$A = F_\beta.$$

F cannot be Borel: Otherwise $D_F = \{\alpha\colon (\alpha,\alpha) \notin F\}$ were Borel. By Corollary 11.3, every Borel set is the image of a closed set under a continuous mapping. This implies that $D_F = F_\beta$. But then

$$\beta \in D_F \quad\Leftrightarrow\quad \beta \in F_\beta \quad\Leftrightarrow\quad (\beta,\beta) \in F \quad\Leftrightarrow\quad \beta \notin D_F,$$

contradiction. □

Lecture 12: Analytic Sets

Definition 12.1: A subset A of a Polish space X is **analytic** if it is empty or there exists a continuous function $f : \mathbb{N}^{\mathbb{N}} \to X$ such that $f(\mathbb{N}^{\mathbb{N}}) = A$.

We will later see that the analytic sets correspond to the sets definable by means of Σ^1_1 formulas, that is formulas in the language of second order arithmetic that have *one existential function quantifier*. Therefore, we will denote the analytic subsets of X also by

$$\boldsymbol{\Sigma}^1_1(X).$$

Here are some simple properties of analytic sets.

Proposition 12.2:

(i) Every Borel set is analytic.

(ii) A continuous image of analytic set is analytic.

(iii) Countable unions of analytic sets are analytic.

Proof. (i) This follows directly from Corollary 11.3.

(ii) The composition of continuous mappings is continuous.

(iii) Let A_n be analytic and $f_n : \mathbb{N}^{\mathbb{N}} \to X$ such that $f_n(\mathbb{N}^{\mathbb{N}}) = A_n$. Define $f : \mathbb{N}^{\mathbb{N}} \to X$ by

$$f(m, \alpha) = f_n(\alpha).$$

Then f is continuous and $f(\mathbb{N}^{\mathbb{N}}) = \bigcup_n A_n$. $\square$

We can use our previous results about Borel sets to give various equivalent characterizations of analytic sets.

Proposition 12.3: *For a subset A of a Polish space X, the following are equivalent.*

(i) *A is analytic,*

(ii) *A is empty or there exists a Polish space Y and a continuous $f : Y \to X$ such that $f(Y) = A$,*

(iii) *A is empty or there exists a Polish space Y, a Borel set $B \subseteq Y$ and a continuous $f : Y \to X$ such that $f(B) = A$.*

(iv) *A is the* projection *of a closed set $F \subseteq \mathbb{N}^{\mathbb{N}} \times X$ along $\mathbb{N}^{\mathbb{N}}$,*

(v) A is the projection of a $\boldsymbol{\Pi}^0_2$ set $G \subseteq 2^{\mathbb{N}} \times X$ along $2^{\mathbb{N}}$,

(vi) A is the projection of a Borel set $B \subseteq X \times Y$ along Y, for some Polish space Y.

Proof. (i) $\iff$ (ii): Follows from Theorem 2.6 and Proposition 12.2 (ii).

(ii) $\iff$ (iii): Follows from Corollary 11.3 and Proposition 12.2 (ii).

(i) $\Rightarrow$ (iv): Let $f : \mathbb{N}^{\mathbb{N}} \to X$ be continuous, $f(\mathbb{N}^{\mathbb{N}}) = A$. Then

$$x \in A \quad \iff \quad \exists \alpha\, (\alpha, x) \in \mathrm{Graph}(f),$$

hence A is the projection of the closed set Graph(f) along $\mathbb{N}^{\mathbb{N}}$.

(iv) $\Rightarrow$ (iii): Clear, since projections are continuous.

(iv) $\Rightarrow$ (v): $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to a $\boldsymbol{\Pi}^0_2$ subset of $2^{\mathbb{N}}$. (Exercise!)

(v) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (iii): Obvious. □

The Lusin Separation Theorem

In a course on computability theory one learns that there are *effectively inseparable* disjoint r.e. sets. i.e. disjoint r.e. sets $W, Z \subseteq \mathbb{N}$ for which no recursive set A exists with $W \subseteq A$ and $A \cap Z = \emptyset$.

In contrast to this, disjoint analytic sets can always be separated by a Borel set, they are **Borel separable**.

Theorem 12.4 (Lusin)**:** *Let $A, B \subseteq X$ be disjoint analytic sets. Then there exists a Borel $C \subseteq X$ such that*

$$A \subseteq C \quad \textit{and} \quad B \cap C = \emptyset,$$

Proof. Let $f : \mathbb{N}^{\mathbb{N}} \to A$ and $g : \mathbb{N}^{\mathbb{N}} \to B$ be continuous surjections.

We argue by contradiction. The key idea is: if A and B are Borel inseparable, then, for some $i, j \in \mathbb{N}$, $A_{\langle i \rangle} = f(N_{\langle i \rangle})$ and $B_{\langle j \rangle} = g(N_{\langle j \rangle})$ are Borel inseparable.

This follows from the observation

($*$) if the sets $R_{m,n}$ separate the sets P_m, Q_n (for each m, n), then $R = \bigcup_m \bigcap_n R_{m,n}$ separates the sets $P = \bigcup_m P_m$, $Q = \bigcup_n Q_n$.

So, by using $(*)$ repeatedly, we can construct sequences $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ such that for all n, $A_{\alpha|_n}$ and $B_{\beta|_n}$ are Borel inseparable, where

$$A_\sigma = f(N_\sigma) \quad \text{and} \quad B_\sigma = g(N_\sigma).$$

Then we have $f(\alpha) \in A$ and $g(\beta) \in B$, and since A and B are disjoint, $f(\alpha) \neq g(\beta)$. Let U, V be disjoint open sets such that $f(\alpha) \in U$, $g(\beta) \in V$. Since f and g are continuous, there exists N such that $f(N_{\alpha|_N}) \subseteq U$, $g(N_{\beta|_N}) \subseteq V$, hence U separates $A_{\alpha|_N}$ and $B_{\beta|_N}$, contradiction. □

The Separation Theorem yields a nice characterization of the Borel sets.

Theorem 12.5 (Souslin)**:** *If a set A and its complement $\neg A$ are both analytic, then A is Borel.*

Proof. In Theorem 12.4, chose $A_0 = A$ and $A_1 = \neg A$. □

Sets whose complement is analytic are called **co-analytic**. Analogous to the levels of the Borel hierarchy, the co-analytic subsets of a Polish space X are denoted by

$$\mathbf{\Pi}^1_1(X).$$

If we define, again analogy to the Borel hierarchy,

$$\mathbf{\Delta}^1_1(X) = \mathbf{\Sigma}^1_1(X) \cap \mathbf{\Pi}^1_1(X),$$

then Souslin's Theorem states that

$$\mathrm{Borel}(X) = \mathbf{\Delta}^1_1(X).$$

The Souslin operation

Souslin schemes give an alternative presentation of analytic sets which will be useful later.

Definition 12.6: A **Souslin scheme** on a Polish space X is a family $P = (P_\sigma)_{\sigma \in \mathbb{N}^{<\mathbb{N}}}$ of subsets of X indexed by $\mathbb{N}^{<\mathbb{N}}$.

The **Souslin operation** $\mathcal{A}$ for a Souslin scheme is given by

$$\mathcal{A}P = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} P_{\alpha|_n}.$$

This means

$$x \in \mathcal{A}P \quad \Longleftrightarrow \quad \exists \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall n \in \mathbb{N}\ x \in P_{\alpha \restriction n}. \tag{$*$}$$

The analytic sets are precisely the sets that can be obtained by Souslin operations on closed sets. If a Γ is a class of sets in various Polish spaces, we let

$$\mathcal{A}\Gamma = \{\mathcal{A}P\colon\ P = (P_\sigma) \text{ is a Souslin scheme with } P_\sigma \in \Gamma \text{ for all } \sigma\}.$$

Theorem 12.7:

$$\mathbf{\Sigma}^1_1(X) \ = \ \mathcal{A}\,\mathbf{\Pi}^0_1(X).$$

Proof. Suppose $f : \mathbb{N}^{\mathbb{N}} \to X$ is continuous with $f(\mathbb{N}^{\mathbb{N}}) = A$. Then

$$x \in A \quad \Leftrightarrow \quad \exists \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall n \in \mathbb{N}\ x \in \overline{f(N_{\alpha|_n})}.$$

Hence if we let $P_\sigma = \overline{f(N_\sigma)}$, then

$$A = \mathcal{A}\,P,$$

for the Souslin scheme $P = (P_\sigma)$.

To see that any set A in $\mathcal{A}\,\mathbf{\Pi}^0_1(X)$ is analytic, consider $(*)$. If the P_σ are closed, the condition

$$(\alpha, x) \in F \quad \Leftrightarrow \quad \forall n \in \mathbb{N}\ x \in P_{\alpha\restriction n}$$

defines a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$ such that A is the projection of F along $\mathbb{N}^{\mathbb{N}}$. $\square$

Note that the Souslin scheme (P_σ) used in the previous proof has the additional property that

$$\sigma \subseteq \tau \quad \Rightarrow \quad P_\sigma \supseteq P_\tau.$$

Such Souslin schemes are called **regular**. By replacing P_σ with

$$Q_\sigma = \bigcap_{\tau \subseteq \sigma} P_\tau,$$

we obtain a Souslin scheme $Q = (Q_\sigma)$ with $\mathcal{A}\,Q = \mathcal{A}\,P$. Moreover, if the P_σ are from a class Γ, and Γ is closed under finite intersections, then the Q_σ are also from Γ. In particular, any analytic set can be obtained from a regular Souslin scheme of closed sets via the Souslin operation.

Lecture 13: Regularity Properties of Analytic Sets

In this lecture we verify that the analytic sets are Lebesgue measurable (LM) and have the Baire property (BP). Since both properties are closed under complements, they also hold for the class of *co-analytic sets* $\mathbf{\Pi}^1_1$.

The analytic sets also have the perfect subset property (PS). As in the case of the Borel sets, the proof uses different ideas and will therefore be presented in a separate lecture. Besides, the perfect subset property for $\mathbf{\Pi}^1_1$ sets is no longer provable in ZF.

For Borel sets, one proves (LM) and (BP) by showing that the class of sets having (LM) (or (BP), respectively) forms a σ-algebra and contains the open sets. For the analytic sets, this method is no longer available. We can, however, prove a similar property with respect to the Souslin operation $\mathcal{A}$, which can be seen as an extension of basic set theoretic operations into the uncountable.

More specifically, we will achieve the following.

- *Show that the Souslin operation $\mathcal{A}$ is **idempotent**, i.e.* $\mathcal{A}\mathcal{A}\,\Gamma = \mathcal{A}\Gamma$. This implies that the analytic sets are closed under $\mathcal{A}$.
- *Show that the family of sets with (LM) (or (BP), respectively), is closed under the Souslin operation.* Since the closed sets have both properties, and the Souslin operator is clearly monotone on classes, this yields the desired regularity results.

Idempotence of the Souslin operation

Theorem 13.1: *For every class Γ of subsets of various Polish spaces,*

$$\mathcal{A}\mathcal{A}\Gamma = \mathcal{A}\Gamma.$$

Proof. We clearly have $\Gamma \subseteq \mathcal{A}\Gamma$, so that we only need to prove $\mathcal{A}\mathcal{A}\Gamma \subseteq \mathcal{A}\Gamma$.

Suppose $A = \mathcal{A}P$ with $P_\sigma \in \mathcal{A}\Gamma$, that is, $P_\sigma = \mathcal{A}Q_{\sigma,\tau}$ mit $Q_{\sigma,\tau} \in \Gamma$. Then

$$\begin{aligned}
z \in A \quad &\Longleftrightarrow \quad \exists \alpha\, \forall m\, (z \in P_{\alpha|_m}) \\
&\Longleftrightarrow \quad \exists \alpha\, \forall m\, \exists \beta\, \forall n\, (z \in Q_{\alpha|_m, \beta|_n}) \\
&\Longleftrightarrow \quad \exists \alpha\, \exists \beta\, \forall m\, \forall n\, (z \in Q_{\alpha|_m, (\beta)_m|_n}),
\end{aligned}$$

where $(\beta)_m$ denotes the m-th column of β.

Now we contract the two function quantifiers to a single one, using a (computable) homeomorphism $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, and the two universal number quantifiers into a single one using the paring function $\langle .,. \rangle$. Then A can be characterized as

$$z \in A \quad \Longleftrightarrow \quad \exists \gamma \ \forall k (z \in R_{\gamma|_k})$$

where $R_\sigma = Q_{\varphi(\sigma),\psi(\sigma)} \in \Gamma$ for suitable coding functions φ, ψ. □

Corollary 13.2:

$$\mathcal{A}\boldsymbol{\Sigma}^1_1 = \boldsymbol{\Sigma}^1_1.$$

Lebesgue measurability of analytic sets

We start with a lemma that essentially says that we can envelop any set with a smallest (up to measure 0) measurable set.

Lemma 13.3: *For every set $A \subseteq \mathbb{R}$ there exists a set $B \subseteq \mathbb{R}$ so that*

(i) $A \subseteq B$ and B is Lebesgue measurable,

(ii) if B' is such that $A \subseteq B' \subseteq B$ and is Lebesgue measurable, then $B \setminus B'$ has measure 0.

Proof. Suppose first that $\lambda^*(A) < \infty$. For every $n \geq 0$, there exists an open set $O_n \supseteq A$ with $\lambda^*(O_n) = \lambda(O_n) < \lambda^*(A) + 1/n$. Then $B = \bigcap_n O_n$ is measurable, and $\lambda(B) = \lambda^*(A)$. Furthermore, if $A \subseteq B' \subseteq B$, then $\lambda^*(A) \leq \lambda^*(B') \leq \lambda^*(B)$. If B' is also measurable, then

$$\lambda^*(B) = \lambda^*(B \cap B') + \lambda^*(B \setminus B') = \lambda^*(B') + \lambda^*(B \setminus B'),$$

hence $\lambda^*(B \setminus B') = 0$.

If $\lambda^*(A) = \infty$, let $A_n = A \cap [m, m+1)$ for $m \in \mathbb{Z}$. Then $\lambda^*(A_m) \leq 1$, and we can choose $B_m \supseteq A_m$ measurable such that $\lambda^*(B_m) = \lambda^*(A_m)$. Then $B = \bigcup_{m \in \mathbb{Z}} B_m$ has the desired property. □

We now apply the lemma to show that Lebesgue measurability is closed under the Souslin operation. The basic idea is to approximate the local 'branches' of the Souslin operation on a Souslin scheme by measurable sets from outside, in the sense of the lemma. It turns out that the total error we make by this approximation is negligible, and hence the overall result of the Souslin operation differs from a measurable set only by a nullset and hence is measurable.

Proposition 13.4: *The class* **LM** *of all Lebesgue measurable sets* $\subseteq \mathbb{R}$ *is closed under the Souslin operation, i.e.*

$$\mathcal{A}\,\mathbf{LM} \subseteq \mathbf{LM}.$$

Proof. Let $A = (A_\sigma)$ be a Souslin scheme with each A_σ measurable. We can assume that (A_σ) is regular. For each $\sigma \in \mathbb{N}^{<\mathbb{N}}$ we let

$$A^\sigma = \bigcup_{\alpha \supset \sigma} \bigcap_{n \in \mathbb{N}} A_{\alpha|_n} \subseteq A_\sigma.$$

Note that $A^{\langle\varnothing\rangle} = \mathcal{A}A$. By the previous lemma, there exist measurable sets $B^\sigma \supseteq A^\sigma$ so that for every measurable $B \supseteq A^\sigma$, $B^\sigma \setminus B$ is null.

By replacing B^σ with $B^\sigma \cap A_\sigma$, we can further assume $B^\sigma \subseteq A_\sigma$. This makes (B^σ) a regular Souslin scheme.

Now let $C_\sigma = B^\sigma \setminus \bigcup_{n\in\mathbb{N}} B^{\sigma^\frown\langle n\rangle}$. Each C_σ is a nullset, by the choice of the B^σ and the fact that $A^\sigma = \bigcup_{n\in\mathbb{N}} A^{\sigma^\frown\langle n\rangle} \subseteq \bigcup_{n\in\mathbb{N}} B^{\sigma^\frown\langle n\rangle}$. Hence $C = \bigcup_\sigma C_\sigma$ is a nullset, too.

It remains to show that

$$B^{\langle\varnothing\rangle} \setminus C \ \subseteq \ A^{\langle\varnothing\rangle} = \mathcal{A}A,$$

for this implies $B^{\langle\varnothing\rangle} \setminus A^{\langle\varnothing\rangle} \subseteq C$ is null, which in turn implies that $A^{\langle\varnothing\rangle}$ is Lebesgue measurable (since it differs from a measurable set by a nullset).

So let $x \in B^{\langle\varnothing\rangle} \setminus C$. Since $x \notin C_{\langle\varnothing\rangle}$, there is an $\alpha(0)$ with $x \in B^{\langle\alpha(0)\rangle}$.

Given $\alpha\,|_n$ with $x \in B^{\alpha|_n}$, we can choose $\alpha(n)$ so that $x \in B^{\alpha|_{n+1}}$. This is possible because $x \notin C_{\alpha|_n}$. This way we construct $\alpha \in \mathbb{N}^\mathbb{N}$ with

$$x \in \bigcap_n B^{\alpha|_n} \subseteq \bigcap_n A_{\alpha|_n} \subseteq A^{\langle\varnothing\rangle}.$$

$\square$

Corollary 13.5: *Every analytic set is Lebesgue measurable.*

Proof. By the idempotence of $\mathcal{A}$, $\mathcal{A}\mathbf{\Sigma}^1_1 = \mathcal{A}\mathcal{A}\mathbf{\Pi}^0_1 = \mathcal{A}\mathbf{\Pi}^0_1 = \mathbf{\Sigma}^1_1$. On the other hand, we have $\mathcal{A}\mathbf{\Pi}^0_1 \subseteq \mathcal{A}\mathbf{LM} = \mathbf{LM}$, since the Souslin operation is monotone on classes. This yields $\mathbf{\Sigma}^1_1 \subseteq \mathbf{LM}$. $\square$

Universally measurable sets

The previous proof is general enough to work for other kinds of measures on arbitrary Polish spaces.

Given a Polish space X, a **Borel measure** on X is a countably additive set function μ defined on a σ-algebra of the Borel sets in X. A set is μ**-measurable** if it can be represented as a union of a Borel set and a μ-nullset. A measure μ is σ**-finite** if $X = \bigcup_n X_n$, where X_n is μ-measurable with $\mu(X_n) < \infty$. Lebesgue measure is σ-finite Borel measure on the Polish space $\mathbb{R}$.

A set $A \subseteq X$ is **universally measurable** if it is μ-measurable for every σ-finite Borel measure on X.

Theorem 13.6 (Lusin)**:** *In a Polish space, every analytic is universally measurable.*

Baire property of analytic sets

Inspecting the proof of Proposition 13.4, we see that it works for the Baire property as well (with *measure* 0 replaced by *meager*, of course), provided we can prove a Baire category version of Lemma 13.3.

Lemma 13.7: *Let X be a Polish space. For every set $A \subseteq X$ there exists a set $B \subseteq X$ so that*

(i) $A \subseteq B$ and B has the Baire property,

(ii) if $Z \subseteq B \setminus A$ and Z has the Baire property, then Z is meager.

Proof. Let $U_1, U_2, \dots$ be an enumeration of countable base of the topology for X. Given $A \subseteq \mathbb{R}$ set

$$A^* := \{x \in \mathbb{R} \colon \forall i\ (x \in U_i \quad \Rightarrow \quad U_i \cap A \text{ not meager})\}.$$

Note that A^* is closed: If $x \notin A^*$, then there exists i with $x \in U_i$ & $U_i \cap A$ null. If $y \in U_i$, then $y \notin A^*$, since $U_i \cap A$ is null. Hence $U_i \subseteq \neg A^*$.

We have

$$A \setminus A^* = \bigcup \{A \cap U_i : A \cap U_i \text{ meager}\},$$

which is a countable union of meager sets and hence meager.

If we let $B = A \cup A^* = A^* \cup (A \setminus A^*)$, then B is a union of a meager set and a closed set and hence has the Baire property.

Now assume $B' \supseteq A$ has the Baire property. Then $C = B \setminus B'$ has the Baire property, too. Suppose C is not meager, then $U_i \setminus C$ is meager for some i, and hence also $U_i \cap A \subseteq (U_i \setminus C)$. Besides, $U_i \cap C \neq \emptyset$, for otherwise $U_i \subseteq U_i \setminus C$ would be meager. Thus there exists $x \in U_i$ with $x \notin A^*$, which by definition of A^* implies that $U_i \cap A$ is not meager, a contradiction. $\square$

By adapting the proof of Proposition 13.4, we obtain the Baire category version of Proposition 13.4 and hence can deduce that analytic sets have the Baire property.

Proposition 13.8: *In any Polish space X, the class* **BP** *of all sets $\subseteq X$ with the Baire property is closed under the Souslin operation, i.e.*

$$\mathcal{A}\,\mathbf{BP} \subseteq \mathbf{BP}.$$

Lecture 14: The Projective Hierarchy

In Lecture 12 we saw that the analytic sets are not closed under complements, which led us to the introduction of the *co-analytic* sets as a separate class.

We saw analytic sets are projections of closed sets and hence can be written as

$$x \in A \quad \Longleftrightarrow \quad \exists \alpha \in \mathbb{N}^{\mathbb{N}}\, F(\alpha, x),$$

where $F \subseteq \mathbb{N}^{\mathbb{N}} \times X$ is closed. It follows that co-analytic sets can be written in the form

$$x \in A \quad \Longleftrightarrow \quad \forall \alpha \in \mathbb{N}^{\mathbb{N}}\, U(\alpha, x),$$

for some open $U \subseteq \mathbb{N}^{\mathbb{N}} \times X$.

Using quantifier manipulations that allow to switch number and function quantifiers,

$$\begin{aligned} \forall m\, \exists \alpha\, P(m, \alpha) \quad &\Longleftrightarrow \quad \exists \beta\, \forall m\, P(m, (\beta)_m) \\ \exists m\, \forall \alpha\, P(m, \alpha) \quad &\Longleftrightarrow \quad \forall \beta\, \exists m\, P(m, (\beta)_m), \end{aligned}$$

we obtain that both the analytic sets and the co-analytic sets are closed under countable unions and intersections.

We have seen (Proposition 12.2) that the analytic sets are closed under continuous images. Taking continuous images of co-analytic sets, however, leads out of the co-analytic sets.

Using continuous images (or rather, the special case of *projections*), we define the **projective hierarchy**. Recall our notation $\exists^{\mathbb{N}}$ for projection along $\mathbb{N}$, with $\forall^{\mathbb{N}}$ its dual. We denote by $\exists^{\mathbb{N}^{\mathbb{N}}}$ and $\forall^{\mathbb{N}^{\mathbb{N}}}$ projection along $\mathbb{N}^{\mathbb{N}}$ and its dual, respectively.

$$\begin{aligned} \boldsymbol{\Sigma}^1_1(X) &= \exists^{\mathbb{N}^{\mathbb{N}}} \boldsymbol{\Pi}^0_1(X) \\ \boldsymbol{\Pi}^1_n(X) &= \neg \boldsymbol{\Sigma}^1_n(X) \\ \boldsymbol{\Sigma}^1_{n+1}(X) &= \exists^{\mathbb{N}^{\mathbb{N}}} \boldsymbol{\Sigma}^1_1(X) \\ \boldsymbol{\Delta}^1_n(X) &= \boldsymbol{\Sigma}^1_n(X) \cap \boldsymbol{\Pi}^1_n(X) \end{aligned}$$

Hence a set $P \subseteq X$ is

$$\begin{array}{llll}
\boldsymbol{\Sigma}^1_1 & \text{iff} & P(x) \iff \exists \alpha\, F(\alpha, x) & \text{for a closed set } F \subseteq \mathbb{N}^{\mathbb{N}} \times X, \\
\boldsymbol{\Pi}^1_1 & \text{iff} & P(x) \iff \forall \alpha\, F(\alpha, x) & \text{for an open set } G \subseteq \mathbb{N}^{\mathbb{N}} \times X, \\
\boldsymbol{\Sigma}^1_2 & \text{iff} & P(x) \iff \exists \alpha \forall \beta\, G(\alpha, \beta, x) & \text{for an open set } G \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times X, \\
\boldsymbol{\Pi}^1_2 & \text{iff} & P(x) \iff \forall \alpha \exists \beta\, F(\alpha, \beta, x) & \text{for a closed set } F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times X, \\
 & & \vdots &
\end{array}$$

These characterizations clearly indicate a relation between being projective and being definable in second order arithmetic using function quantifiers. We will describe this relation in detail when we address the effective ('lightface') version of the projective hierarchy.

Examples of projective sets

Here are a few examples of projective sets that occur naturally in mathematics.

Analytic sets:

- $\{K \subseteq X : K \text{ compact and uncountable}\}$ is a $\boldsymbol{\Sigma}^1_1$ subset of the space $K(X)$ of compact subsets of X.
- $\{f \in \mathcal{C}[0,1] : f \text{ continuously differentiable on } [0,1]\}$ is a $\boldsymbol{\Sigma}^1_1$ subset of $\mathcal{C}[0,1]$.

Co-analytic sets:

- $\{f \in \mathcal{C}[0,1] : f \text{ differentiable on } [0,1]\}$ is a $\boldsymbol{\Pi}^1_1$ subset of $\mathcal{C}[0,1]$.
- $\{f \in \mathcal{C}[0,1] : f \text{ nowhere differentiable on } [0,1]\}$ is a $\boldsymbol{\Pi}^1_1$ subset of $\mathcal{C}[0,1]$.
- $\mathrm{WF} = \{\alpha \in 2^{\mathbb{N}} : \alpha \text{ codes a well-founded tree on } \mathbb{N}\}$ is a $\boldsymbol{\Pi}^1_1$ subset of the space Tr of trees, which can be seen as a closed subspace of $2^{\mathbb{N}^{<\mathbb{N}}}$, and hence is Polish. As we will see, the set WF is a prototypical $\boldsymbol{\Pi}^1_1$ set.

Higher levels:

- $\{f \in \mathcal{C}[0,1] : f \text{ satisfies the Mean Value Theorem } [0,1]\}$ is a $\boldsymbol{\Pi}^1_2$ subset of $\mathcal{C}[0,1]$.

  (Here f satisfies the Mean Value Theorem if for all $a < b \in [0,1]$ there exists c with $a < c < b$ such that $f'(c)$ exists and $f(b) - f(a) = f'(c)(b - a)$.)

The quantifier manipulations mentioned above yield the following closure properties.

Proposition 14.1:

(1) *The classes $\boldsymbol{\Sigma}^1_n$ are closed under continuous preimages, countable intersections and unions, and continuous images (in particular, $\exists^{\mathbb{N}^{\mathbb{N}}}$).*

(2) *The classes $\boldsymbol{\Pi}^1_n$ are closed under continuous preimages, countable intersections and unions, and co-projections $\forall^{\mathbb{N}^{\mathbb{N}}}$.*

(3) *The classes $\boldsymbol{\Delta}^1_n$ are closed under continuous preimages, complements, countable intersections and unions. (In particular, they for a σ-algebra.)*

To show that the hierarchy is proper, we need the existence of universal sets.

Proposition 14.2: *For every Polish space X, there is a $\mathbb{N}^{\mathbb{N}}$-universal set for $\boldsymbol{\Sigma}^1_n$ and for $\boldsymbol{\Pi}^1_n$.*

Proof. By induction on n. We have seen that there exists a $\mathbb{N}^{\mathbb{N}}$-universal set for $\boldsymbol{\Sigma}^1_1$. Now note that if $U \in \boldsymbol{\Sigma}^1_n(\mathbb{N}^{\mathbb{N}} \times X)$ is $\mathbb{N}^{\mathbb{N}}$-universal for $\boldsymbol{\Sigma}^1_n(X)$, then $\neg U$ is $\mathbb{N}^{\mathbb{N}}$-universal for $\boldsymbol{\Pi}^1_n(X)$, and if $U \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times X$ is $\mathbb{N}^{\mathbb{N}}$-universal for $\boldsymbol{\Pi}^1_n(\mathbb{N}^{\mathbb{N}} \times X)$, then

$$V = \{(\alpha, z) \colon \exists \beta \, (\alpha, \beta, z) \in U\}$$

is $\mathbb{N}^{\mathbb{N}}$-universal for $\boldsymbol{\Sigma}^1_{n+1}$. □

Corollary 14.3: *For every $n \geq 1$, $\boldsymbol{\Sigma}^1_n \nsubseteq \boldsymbol{\Pi}^1_n$ and $\boldsymbol{\Pi}^1_n \nsubseteq \boldsymbol{\Sigma}^1_n$. Morever,*

$$\begin{aligned} \boldsymbol{\Sigma}^1_n &\subsetneq \boldsymbol{\Delta}^1_{n+1} \subsetneq \boldsymbol{\Sigma}^1_{n+1} \\ \boldsymbol{\Pi}^1_n &\subsetneq \boldsymbol{\Delta}^1_{n+1} \subsetneq \boldsymbol{\Pi}^1_{n+1} \end{aligned}$$

The proof is similar to the proofs of Theorem 10.7 and Corollary 10.8.

Regularity properties of projective sets

At first sight it does not seem impossible to extend the regularity properties (LM) and (BP) to higher levels of the projective hierarchy. But we will soon see that there are metamathematical limits that prevent us from doing so.

Without explicitly mentioning it, up to now we have been working in ZF, Zermelo-Fraenkel set theory, plus a weak form of Choice ($\mathsf{AC}_\omega(\mathbb{N}^\mathbb{N})$). If we add the full Axiom of Choice (AC), we saw that the regularity properties do not extend to all sets. Solovay's model of ZF showed that the use of a strong version of Choice is necessary for this.

On the other hand, the proofs gave us no direct indication how 'complex' the non-regular sets we constructed are. In the next section we will start to study a model of ZF in which exists a $\boldsymbol{\Delta}^1_2$ set which is neither Lebesgue measurable nor does it have the Baire property. Therefore, we cannot settle in ZF the question of whether the projective sets are measurable or have the Baire property. We will have to add additional axioms.

A key feature in the construction of a non-measurable $\boldsymbol{\Delta}^1_1$ set is the use of the well-ordering principle rather than the Axiom of Choice.

Proposition 14.4: *Suppose* $<_W \subseteq \mathbb{R} \times \mathbb{R}$ *is a well-ordering of* $\mathbb{R}$ *of order-type* ω_1 *in* Γ*, then there exists a subset of* $\mathbb{R}$ *in* Γ *that is neither Lebesgue measurable nor has the Baire property.*

Lebesgue measure here refers to the product measure $\lambda \times \lambda$, which is the unique translation invariant measure defined on the Borel σ-algebra generated by the rectangles $I \times J$, where I and J are open intervals, and $(\lambda \times \lambda)(I \times J) = \lambda(I)\lambda(J)$.

Proof. Suppose $<_W$ is a well-ordering of $\mathbb{R}$ in Γ. Let $A = \{(x, y) \colon x <_W y\}$.

Since $<_W$ is of order type ω_1, for every $y \in \mathbb{R}$, the set $A_y = \{x \colon x <_W y\}$ is countable, and hence of Lebesgue measure zero.

Fubini's Theorem implies that if $A \subseteq \mathbb{R}^2$ is measurable, then

$$(\lambda \times \lambda)(A) = \int \lambda(A_y) d\lambda(y) = 0.$$

So if A is measurable, then $(\lambda \times \lambda)(A) = 0$. The complement of A is $\neg A = \{(x, y) \colon x \geq_W y\}$. As above, for any $x \in \mathbb{R}$, $\neg A_x = \{y \colon x \leq_W y\}$ is countable, and hence $\lambda(\neg A_x) = 0$ for all x. Again, by Fubini's Theorem, $(\lambda \times \lambda)(\neg A) = 0$, and thus $(\lambda \times \lambda)(\mathbb{R}) = (\lambda \times \lambda)(A \cup \neg A) = (\lambda \times \lambda)(A) + (\lambda \times \lambda)(\neg A) = 0$, a contradiction.

We can apply a similar reasoning for Baire category. The sections A_y and $\neg A_x$ are countable, and hence meager.

The following lemma provides a Baire category analogue to Fubini's Theorem.

Lemma 14.5: *Let $A \subseteq \mathbb{R}^2$ have the property of Baire. Then A is meager if and only if $A_x = \{y : (x, y) \in A\}$ is meager for all x except a meager set.*

For a proof see **(author?)** [Kec95]. $\square$

Therefore, if the Continuum hypothesis (CH) holds in a model and we can well-order $\mathbb{R}$ (or $\mathbb{N}^{\mathbb{N}}$, $2^{\mathbb{N}}$) within a certain complexity (as a subset of $\mathbb{R}^2$), we can find a non-regular set of the same complexity. The question now becomes how (hard it is) to define a well-ordering of $\mathbb{R}$, and of course if CH holds.

Lecture 15: The Constructible Universe

A set X is *first-order definable* in a set Y (from parameters) if there exists a first-order formula $\varphi(x_0, x_1, \dots, x_n)$ in the language of set theory (i.e. only using the binary relation symbol $\in$) such that for some $a_1, \dots, a_n \in Y$,

$$X = \{y \in Y : (Y, \in) \models \varphi[y, a_1, \dots, a_n]\}.$$

Here $(Y, \in)$ stands for the interpretation of Y as a structure of the language of set theory, i.e. Y is a set and $\in$ is interpreted as a binary relation over Y.

The constructible universe is built as a cumulative hierarchy of sets along the ordinals. In each successor step, instead of adding all subsets of the current set, only the *definable* ones are added. Formally, L is defined as follows. Given a set Y, let

$$\mathcal{P}_{\mathrm{DEF}}(Y) = \{X \subseteq Y : X \text{ is first order definable in } Y \text{ from parameters}\},$$

where the underlying language is the language of set theory. Now put

$$\begin{aligned} L_0 &= \emptyset \\ L_{\xi+1} &= \mathcal{P}_{\mathrm{DEF}}(L_\xi) \\ L_\xi &= \bigcup_{\zeta<\xi} L_\zeta \quad (\xi \text{ limit ordinal}) \end{aligned}$$

Finally, let

$$L = \bigcup_{\xi \in \mathrm{Ord}} L_\xi.$$

Basic properties of L

The first Proposition tells us that the L_ξ are set-theoretically nice structures and linearly ordered by the $\subseteq$-relation.

Proposition 15.1: *For each ordinal ξ:*

(1) L_ξ is transitive.

(2) For $\zeta < \xi$, $L_\zeta \subseteq L_\xi$.

(3) For $\zeta < \xi$, $L_\zeta \in L_\xi$.

(4) *If* $\xi \geq \omega$*, then* $|L_{\xi+1}| = |L_\xi|$.

Proof. We show the first two statements statements simultaneously by induction. They are clear for $\xi = 0$ and ξ limit, so assume $\xi = \zeta + 1$. Suppose $x \in L_\zeta$. Consider the formula $\varphi(x_0) \equiv x_0 \in x$ (here x is a parameter). φ defines the set

$$x' = \{a \in L_\zeta : L_\zeta \models \varphi[a]\} = \{a \in L_\zeta : a \in x\}.$$

By induction hypothesis, L_ζ is transitive, and hence $a \in x$ implies $a \in L_\zeta$, and hence $x' = x$, so $x \in L_{\zeta+1}$. This yields $L_\zeta \subseteq L_\xi$. Now if $x \in L_\xi$, then $x \subseteq L_\zeta$, and hence $x \subseteq L_\xi$. Thus L_ξ is transitive.

For the third statement, note that the formula $\varphi(x_0) \equiv x_0 = x_0$ defines L_ζ in L_ζ, and hence $L_\zeta \in L_{\zeta+1}$. □

Next, we show that L contains all ordinals and that ξ 'shows up' exactly after ξ steps.

Proposition 15.2: *For any* ξ*,*

(1) $\xi \subseteq L_\xi$*,*

(2) $L_\xi \cap \mathrm{Ord} = \xi$.

Proof. Clearly, (1) follows from (2). To show (2), one again proceeds by induction. Again, the statement is clear for 0 and limit ordinals, so assume $\xi = \zeta + 1$ and $L_\zeta \cap \mathrm{Ord} = \zeta$. We need to show $L_{\zeta+1} \cap \mathrm{Ord} = \zeta + 1 = \zeta \cup \{\zeta\}$. Since $L_\zeta \subseteq L_{\zeta+1}$, we have $\zeta \subseteq L_{\zeta+1} \cap \mathrm{Ord}$. On the other hand, since $L_{\zeta+1} \subseteq \mathcal{P}(L_\zeta)$, we have $L_{\zeta+1} \cap \mathrm{Ord} \subseteq \zeta + 1$. It thus remains to show that $\zeta \in L_{\zeta+1}$.

We need a formula φ_{Ord} that defines the ordinals (in L_ζ). Such is formula is easily found by formalizing the statement

"x is transitive and linearly ordered by $\in$*."*

(Note that we assume that every set is well-founded.) It then seems that we have

$$\zeta = \{a \in L_\zeta : L_\zeta \models \varphi_{\mathrm{Ord}}[a]\}, \tag{$*$}$$

and hence we can conclude $\zeta \in L_{\zeta+1}$. The problem is that *being an ordinal* (i.e. satisfying φ_{Ord}) *in* L_ζ may not be the same as *being an ordinal in general* (in V, the universe of all sets).

The fact that $(*)$ nevertheless is true is a consequence of the *absoluteness of* Δ_0 *formulas for transitive sets*. We address this important concept in detail next. □

Given a formula φ in the language of set theory and some class M, we can *relativize* φ to φ^M essentially by restricting all quantifiers occurring in φ to range over M, i.e. $\exists x\, \psi$ becomes $(\exists x \in M)\, \psi^M$, for example. We say a formula $\varphi(x_0, \dots, x_n)$ is **absolute for** M if for all $a_0, \dots, a_n \in M$

$$\varphi^M(a_0, \dots, a_n) \text{ holds} \quad \Longleftrightarrow \quad \varphi(a_0, \dots, a_n) \text{ holds.}$$

Unfortunately, even simple formulas like $x \subseteq y$ can fail to be absolute. For example, let $M = \{0, a\}$, where $a = \{\{0\}\}$. Then $(a \subseteq 0)^M$ (which is defined as $\forall x \in M\, (x \in a \ \to\ x \in 0)$) but not $a \subseteq 0$.

However, if M is transitive, then many important formulas *are* absolute for M. A formula is Δ_0 if it contains no or only *bounded quantifiers* of the form $\forall x \in v$ or $\exists x \in v$, where x, v are set variables.

Proposition 15.3: *If M is transitive and φ is Δ_0, then φ is absolute for M.*

Proof sketch. Clearly $x = y$ and $x \in y$ are absolute for any M. It is also not hard to see that if φ and ψ are absolute for M, then so are $\neg\varphi$ and $\varphi \wedge \psi$. Hence all quantifier free formulas are absolute.

Finally, if φ is absolute for M, so is $\psi \equiv \exists x \in y\, \varphi$: If $\psi^M(y, \bar{z})$ holds for $y, \bar{z} \in M$, then we have $[\exists x (x \in y \ \wedge\ \varphi(x, y, \bar{z}))]^M$, i.e., $\exists x \in M (x \in y \ \wedge\ \varphi^M(x, y, \bar{z}))$. Since $\varphi^M(x, y, \bar{z})$ if and only if $\varphi(x, y, \bar{z})$, it follows that $\exists x \in y \varphi(x, y, \bar{z})$, i.e. ψ.

Conversely, if for $y, \bar{z} \in M$, $\exists x \in y \varphi(x, y, \bar{z})$, then since M is transitive, x belongs to M, and since $\varphi(x, y, \bar{z})$ if and only $\varphi^M(x, y, \bar{z})$, we have $\exists x \in M\, (x \in y \ \wedge\ \varphi^M(x, y, \bar{z}))$ and so $\psi^M(y, \bar{z})$. $\square$

On can show that "*x is an ordinal*." is indeed definable by a Δ_0 formula. Furthermore, using the absoluteness of Δ_0 formulas, one can also show that L is a model of ZF. More formally, this means that for every axiom σ of ZF, $\mathsf{ZF} \vdash \sigma^L$.

Theorem 15.4: *For every axiom σ of* ZF, $\mathsf{ZF} \vdash \sigma^L$.

L is an **inner model** of ZF, that is, L is transitive, contains all ordinals, and satisfies the axioms of ZF.

We can add to ZF the axiom that all sets are constructible, i.e.

$$\forall x \exists y\, (y \text{ is an ordinal} \ \wedge\ x \in L_y).$$

This axiom is usually denoted by $\mathsf{V} = \mathsf{L}$. We may be tempted to think that L is then trivially a model of $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$. But this is not at all clear, since this has to hold *relative to* L, i.e. $(\mathsf{V} = \mathsf{L})^L$. This means that

$$\forall x \in L \, \exists y \in L \, (y \text{ is an ordinal } \wedge \, (x \in L_y)^L).$$

But it might be that $(x \in L_y)^L$ is not absolute, that is, viewed from *inside* L, not every set may be definable. To show that $(x \in L_y)^L$ is indeed absolute, one has to carefully study the notion of definability. In particular, we have to show that *definability is definable*.

The definability of L

Fix a Gödel numbering of set theoretic formulae. We can use it to formally define syntactical notions such as the satisfaction relation. More precisely, given a set X, let $\mathrm{SAT}_X : X^{<\omega} \times \omega \to \{0,1\}$ be the binary valued (partial) function that is defined for $((a_1, \dots, a_n), e)$ iff e is the Gödel number of a formula φ with n free variables and in this case

$$\mathrm{SAT}_X(\vec{a}, e) = 1 \quad \text{iff} \quad X \models \varphi[\vec{a}]. \tag{15.1}$$

While Tarski's Theorem excludes the possibility that a structure X satisfying (a sufficiently large fragment of) ZFC can define its own truth predicate, one can formalize the satisfaction relation and show it works in a *relativized* environment that has sufficient closure properties. To be more precise, based on the recursive definition of the satisfaction relation one can devise a set theoretic formula φ_{SAT} aiming at describing this relation formally. This formula will "work" in any relativized environment, represented by a set Y, as long as Y satisfies some basic closure properties – it has to be transitive, closed under formation of finite sequences, and has to be able to address the Gödel numbers of formulas, i.e. it contains the natural numbers. The latter can be ensured by requiring that $V_\omega \subseteq Y$. Let us call such Y *adequate*.

Proposition 15.5: *There exists a set theoretic formula $\varphi_{\mathrm{SAT}}(x_0, x_1)$ such that for all adequate Y, whenever $a_0, a_1 \in Y$,*

$$Y \models \varphi_{\mathrm{SAT}}[a_0, a_1] \quad \textit{iff} \quad \begin{array}{l} (1)\ a_0 \textit{ is transitive and} \\ (2)\ a_1 = \mathrm{SAT}_{a_0}. \end{array}$$

Based on φ_{SAT}, one can devise a formula φ_{DEF} with the following properties: Suppose Y is adequate. Then for all $a_0, a_1 \in Y$, if $\mathrm{SAT}_{a_0} \in Y$ then

$$Y \models \varphi_{\mathrm{DEF}}[a_0, a_1] \quad \text{iff} \quad \begin{array}{l} (1)\ a_0 \text{ is transitive and} \\ (2)\ a_1 = \mathcal{P}_{\mathrm{DEF}}(a_0). \end{array}$$

In other words, first-order definability is definable. With regard to absoluteness considerations, it is important to track the complexity of the formulas. It turns out φ_{DEF} is provably equivalent in ZF to both a Σ_1 and a Π_1 formula of set theory. In this case we say the predicate $a_1 = \mathcal{P}_{\mathrm{DEF}}(a_0)$ is Δ_1.

It is not hard to see that for limit ξ, L_ξ is adequate. One can use these closure properties of L_ξ at limit stages to show that L_ξ can definably "recover" the sequence of L_ζ's leading up to it.

Proposition 15.6: *Suppose ξ is a limit ordinal, $\xi > 0$. Let $G : \mathrm{Ord} \to V$ be given by $\zeta \mapsto L_\zeta$. Then for all $\zeta < \xi$, $G|_\zeta \in L_\xi$.*

If a formula $\varphi(\bar{x}, y)$ defines a function $F(\bar{x}) = y$, then we say F is *absolute for M* if φ is. (It is not hard to show that this is independent of the particular definition of F.) One can in fact show that the function G is absolute for all transitive models of ZF (it is Δ_1).

Theorem 15.7: *L is a model of $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$.*

Proof. If $x \in L$, then there exists a limit ξ such that $x \in L_\xi$. Since $\mathrm{Ord} \subseteq L$, and since G is absolute,

$$\begin{aligned} \forall x \in L \exists \xi \in L (x \in L_\xi) \quad &\Leftrightarrow \quad \forall x \in L \exists \xi, y \in L (G(\xi) = y \ \wedge \ x \in y) \\ &\Leftrightarrow \quad \forall x \in L \exists \xi \in L\, [(x \in L_\xi)^L]. \end{aligned}$$

$\square$

A further consequence is that L is the *smallest* transitive class model of ZF.

Theorem 15.8: *If M is any transitive proper class model of ZF, then $L = L^M \subseteq M$.*

The crucial fact used to prove Proposition 15.6 is that for transitive $X \in L_\xi$,

$$\text{SAT}_X \text{ can be "reached" from } X \text{ within a \textit{finite} number of iterations of the } \mathcal{P}_{\mathrm{DEF}}\text{-operator.} \quad (15.2)$$

From this it follows that $\mathrm{SAT}_X \in L_\xi$, and hence L_ξ is closed under the SAT-function.

Note that the proposition is *not* an immediate consequence of the definition of L. Although we have that $L_\zeta \in L_\xi$ for all $\zeta < \xi$, it is not clear at all that in L_ξ one can define the whole ensemble of the L_ζ in first order terms. The proposition says that we can *definably* recover them in L_ξ: There is a formula $\varphi_L(x_0, x_1)$ such that for limit ξ, given $a_0, a_1 \in L_\xi$,

$$L_\xi \models \varphi_L[a_0, a_1] \quad \text{iff} \quad a_0 \text{ is an ordinal and } a_1 = L_{a_0}.$$

As a consequence, one can devise a sentence $\varphi_{\mathsf{V=L}}$ that identifies precisely the limit levels of the constructible hierarchy: For any transitive set Y,

$$Y \models \varphi_{\mathsf{V=L}} \quad \text{iff} \quad Y = L_\xi \text{ for some limit ordinal } \xi.$$

This last fact has a far-reaching implication.

Theorem 15.9 (Gödel Condensation Lemma)**:** *For every limit ordinal ζ, every elementary substructure of $(L_\zeta, \in)$ is isomorphic to an $(L_\eta, \in)$ for some $\eta \leq \zeta$.*

The canonical well-ordering of L

Every well-ordering on a transitive set X can be extended to a well-ordering of $\mathcal{P}_{\mathrm{DEF}}(X)$. Note that every element of $\mathcal{P}_{\mathrm{DEF}}(X)$ is determined by a pair $(\psi, \vec{a})$, where ψ is a set-theoretic formula, and $\vec{a} = (a_1, \dots, a_n) \in X^{<\omega}$ is a finite sequence of parameters. For each $z \in \mathcal{P}_{\mathrm{DEF}}(X)$ there may exist more than one such pair (i.e. z can have more than one definition), but by well-ordering the pairs $(\psi, \vec{a})$, we can assign each $z \in \mathcal{P}_{\mathrm{DEF}}(X)$ its *least* definition, and subsequently order $\mathcal{P}_{\mathrm{DEF}}(X)$ by comparing least definitions. Elements already in X will form an initial segment. Such an order on the pairs $(\psi, \vec{a})$ can be obtained in a definable way: First use the order on X to order $X^{<\omega}$ length-lexicographically, order the formulas through their Gödel numbers, and finally say

$$(\psi, \vec{a}) < (\varphi, \vec{b}) \quad \text{iff} \quad \psi < \varphi \text{ or } (\psi < \varphi \text{ and } \vec{a} < \vec{b}).$$

Based on this, we can order all levels of L so that the following hold:

(1) $<_L |V_\omega$ is the canonical well-order on V_ω.

(2) $<_L |L_{\zeta+1}$ is the order on $\mathcal{P}_{\mathrm{DEF}}(L_\zeta)$ induced by $<_L |L_\zeta$.

(3) $<_L |L_\zeta = \bigcup_{\zeta<\xi} <_L |L_\zeta$ for a limit ordinal $\xi > \omega$.

It is straightforward to verify that this is indeed a well-ordering on L. But more importantly, for any limit ordinal $\xi > \omega$, $<_L |L_\xi$ is definable over L_ξ. To facilitate notation, we denote the restriction of $<_L$ to some L_ξ by $<_\xi$.

Proposition 15.10: *There is a Σ_1 formula $\varphi_<(x_0, x_1)$ such that for all limit ordinals $\xi > \omega$, if $a, b \in L_\xi$,*

$$L_\xi \models \varphi_<[a, b] \quad \text{iff} \quad a <_\xi b.$$

The proof of this proposition is similar to the proof that the sequence of $(L_\zeta)_{\zeta<\xi}$ is definable in L_ξ. It relies on the strong closure properties of L_ξ under the SAT-function.

Theorem 15.11: *If* $\mathsf{V} = \mathsf{L}$ *then* AC *holds.*

The Continuum Hypothesis in L

We show that the Generalized Continuum Hypothesis (GCH) holds if $\mathsf{V} = \mathsf{L}$.

Theorem 15.12 (Gödel)**:** *If* $\mathsf{V} = \mathsf{L}$*, then for any ordinal* ξ*,* $2^{\aleph_\xi} = \aleph_{\xi+1}$.

Proof sketch. Suppose $A \subseteq L \cap \aleph_\xi$. Since we assume $\mathsf{V} = \mathsf{L}$, there exists limit $\delta > \aleph_\xi$ such that $A \in L_\delta$. Let $X = \aleph_\xi \cup \{A\}$. By choice of δ, $X \subseteq L_\delta$. The Löwenheim-Skolem Theorem (and a Mostowski collapse – see Lecture 16) yields a set M such that

- $(M, \in)$ is a transitive, elementary substructure of $(L_\delta, \in)$,
- $X \subseteq M \subseteq L_\delta$,
- $|M| = |X|$.

The Condensation Lemma 15.9 yields that $M = L_\zeta$ for some $\zeta \leq \delta$.

Lemma 15.13: *For all $\xi \geq \omega$, $|L_\xi| = |\xi|$.*

Proof of Lemma. We know that $\xi \subseteq L_\xi$. Hence $|\xi| \leq |L_\xi|$. To show $|\xi| \geq |L_\xi|$, we work by induction on ξ.

If $\xi = \delta + 1$, then by Proposition 15.1 (4), $|L_\xi| = |L_\delta| = |\delta| \leq |\xi|$.

If ξ is limit, then L_ξ is a union of $|\xi|$ many sets of cardinality $\leq |\xi|$ (by inductive hypothesis), hence of cardinality $\leq |\xi|$. $\square$

Applying the lemma to $M = L_\zeta$, we obtain

$$|\zeta| = |L_\zeta| = |M| = |X| = \aleph_\xi < \delta.$$

Therefore, $A \subseteq L_\zeta$, $|\zeta| < \aleph_\xi$, which means that every subset of $L \cap \aleph_\xi$ appears (is constructed) at an ordinal $< \aleph_{\xi+1}$, and therefore $L \cap \mathcal{P}(\aleph_\xi) \subseteq L_{\aleph_{\xi+1}}$, and hence, by the Lemma,

$$|L \cap \mathcal{P}(\aleph_\xi)| \leq |L_{\aleph_{\xi+1}}| = \aleph_{\xi+1}.$$

□

In the previous proof we have used the Axiom of Choice in various places (Löwenheim-Skolem, proof of the lemma), but since $\mathsf{V} = \mathsf{L}$ implies AC, this is not a problem.

Lecture 16: Constructible Reals

In this lecture we transfer the results about L to the projective hierarchy. The main idea is to relate sets of reals that are defined by set theoretic formulas to sets defined in second order arithmetic.

The effective projective hierarchy

We have seen that the Borel sets of finite order correspond to the sets definable (from parameters) by formulas using only number quantifiers (*arithmetical formulas*). A similar relation holds between projective sets and sets definable by formulas using both number and function quantifiers. In fact, the way we defined the projective hierarchy makes this easy to see.

Historically, however, the topological approach and the definability approach happened separably, the former devised by the Russian school of Souslin, Lusin, and others, while the effective approach was pursued by Kleene. Kleene named the sets definable over second order arithmetic the *analytical* sets, which to this day is a source of much confusion.

Definition 16.1 (Kleene)**:** A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ^1_n if there exists an arithmetical formula $\varphi(\alpha, \beta_1, \dots, \beta_n)$ such that

$$\alpha \in A \quad \Leftrightarrow \quad \exists \beta_1 \forall \beta_2 \dots Q\beta_n \, \varphi(\alpha, \beta_1, \dots, \beta_n)$$

where Q is $\exists$ if n is odd and Q is $\forall$ if n is even. Similarly, $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^1_n if there exists an arithmetical formula $\varphi(\alpha, \beta_1, \dots, \beta_n)$ such that

$$\alpha \in A \quad \Leftrightarrow \quad \forall \beta_1 \exists \beta_2 \dots Q\beta_n \, \varphi(\alpha, \beta_1, \dots, \beta_n)$$

where Q is $\forall$ if n is odd and Q is $\exists$ if n is even. A set that is Σ^1_n and Π^1_n at the same time is called Δ^1_n. A set A is **analytical** if it is Σ^1_n or Π^1_n.

It is often useful only having to deal with analytical predicates of certain form. Kleene provided normal forms of analytical predicates.

Proposition 16.2 (Kleene)**:** *Every analytical predicate $A(\alpha)$ is equivalent to one of the following forms:*

$$\begin{array}{lccc} & \exists \beta \forall m \psi(\alpha, \beta, m) & \exists \beta \forall \gamma \exists m \psi(\alpha, \beta, \gamma, m) & \dots \\ \theta(\alpha, \beta, \gamma, m) & & & \\ & \forall \beta \exists m \psi(\alpha, \beta, m) & \forall \beta \exists \gamma \forall m \psi(\alpha, \beta, \gamma, m) & \dots \end{array}$$

where θ is arithmetic, and ψ is a formula whose quantifiers (if any) are bounded number quantifiers.

The Normal Form is proved by applying a sequence of quantifier manipulations. We provide a sufficient list:

$$
\begin{array}{rcl}
\forall m\,\exists\alpha\,\varphi(\alpha,m) & \Leftrightarrow & \exists\alpha\,\forall m\,\varphi((\alpha)_m,m)\\
\exists m\,\varphi(m) & \Leftrightarrow & \exists\alpha\,\varphi(\alpha(0))\\
\exists\alpha\,\exists\beta\,\varphi(\alpha,\beta) & \Leftrightarrow & \exists\gamma\,\varphi((\gamma)_0,(\gamma)_1)\\
\exists m\,\exists n\,\varphi(m,n) & \Leftrightarrow & \exists k\,\varphi((k)_0,(k)_1)
\end{array}
$$

Each rule has a dual obtained by flipping the quantifiers. The quantifier manipulations can also be used to show

Proposition 16.3 (Kleene)**:** *A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is definable over second order arithmetic if it is analytical.*

The following theorem complements Theorem 9.9. It is an immediate consequence of the definition of the classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

Theorem 16.4: *A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Sigma}^1_n$ ($\mathbf{\Pi}^1_n$) if and only if it is definable in γ by a Σ^1_n (Π^1_n) formula, for some $\gamma \in \mathbb{N}^{\mathbb{N}}$.*

The set of constructible reals

What is the complexity of the set $\mathbb{N}^{\mathbb{N}} \cap L$? In particular, is it in the projective hierarchy? The set of all constructible reals is defined by a Σ_1 formula over set theory:

$$\varphi(x_0)\exists y\ [\,y \text{ is an ordinal } \wedge\ x_0 \in L_y\ \wedge\ x_0 \text{ is a set of natural numbers }].$$

We would like to replace this formula by an "equivalent" one in the language of second order arithmetic. In particular, we would like to replace the quantifier $\exists y$ by a quantifier over a real number.

The key for doing this is the fact that every constructible real shows up at a countable stage of L: Since CH holds in L, $L \cap \mathcal{P}(\omega) \subseteq L_{\omega_1}$. Hence if $\alpha \in L \cap \mathbb{N}^{\mathbb{N}}$, there exists a countable ξ such that $\alpha \in L_\xi$. Since $|\xi| = |L_\xi|$, L_ξ is countable, too. Hence we can hope to replace L_ξ by something like "*there exists a real that codes a model that looks like* L_ξ".

A set theoretic structure is simply a set X with a binary relation (the interpretation of $\in$). If X is countable (infinite), we can assume $X = \omega$, and then any $\alpha \in \mathbb{N}^{\mathbb{N}}$ codes the set theoretic structure

$$(\omega, E_\alpha) \qquad \text{where } E_\alpha = \{\langle m,n\rangle : \alpha(\langle m,n\rangle) = 0\}.$$

We know from the previous lecture that there exists a sentence $\varphi_{\mathsf{V=L}}$ so that if Y is a transitive set, $Y \models \varphi_{\mathsf{V=L}}$ if and only if $Y = L_\delta$ for some limit δ. But for an arbitrary real α, E_α does not need to look anything like a set. It may even fail to be well-founded as a relation. However, if E_α *is* well-founded and *extensional*, then it looks very much like a (transitive) set.

The Mostowski collapse

Let E be a binary relation on a set X. Think of (X, P) as an intended model of set theory. We would like E to behave like the $\in$-relation for sets. For this purpose, let for each $x \in M$

$$\operatorname{ext}_E(x) = \{y \in X : y\, E\, x\}$$

If E behaves “set-like”, then it will respect the *Axiom of Extensionality*, i.e. two sets are identical if and only if they have the same elements. Therefore we say that E is **extensional** if

$$x, z \in X,\ x \neq z \quad \text{implies} \quad \operatorname{ext}_E(x) \neq \operatorname{ext}_E(z).$$

Furthermore, we want to exclude infinite descending E-chains. We say that E is **well-founded** if

every non-empty set $Y \subseteq X$ has an E-minimal subset.

Theorem 16.5: *If E is an extensional and well-founded relation on a set X, then there exists a transitive set S and a bijection $\pi : X \to S$ such that*

$$x\, E\, y \quad \Longleftrightarrow \quad \pi(x) \in \pi(y) \quad \textit{for all} x, y \in X.$$

Moreover, S and π are unique.

Proof. We construct π and $S = \operatorname{im}(\pi)$ by recursion on E, which is possible since it is well-founded. (For details on recursion and induction on well-founded relations, see **(author?)** [Jec03].) For each $x \in X$, let

$$\pi(x) = \{\pi(y) : y\, E\, x\},$$

and set $S = \mathrm{im}(\pi)$.

The injectivity of π follows from the extensionality of π by induction along E: Suppose we have shown

$$\forall z\ (zEx \to \forall y \in X(\pi(z) = \pi(y) \to z = y)).$$

and we have to show that it holds for x. Assume $\pi(x) = \pi(y)$ for some $y \in X$. Then

$$\begin{aligned} cEx &\to \pi(c) \in \pi(x) = \pi(y) \\ &\to \pi(c) = \pi(z) \text{ for some } zEy \\ &\to c = z \text{ by Ind. Hyp., since } cEx \\ &\to cEy. \end{aligned}$$

Similarly, we get $cEy \to cEx$, hence $x = y$ as desired due to the extensionality of E. Finally we have

$$\begin{aligned} \pi(x) \in \pi(y) &\to \pi(x) = \pi(c) \text{ for some } cEy \\ &\to x = c \text{ since } \pi \text{ is injective} \\ &\to xEy. \end{aligned}$$

Thus π is an isomorphism.

To see the uniqueness of π and S, assume ρ, T are such that the statement of the Theorem is satisfied. Then $\pi \circ \rho^{-1}$ is an isomorphism between $(T, \in)$ and $(S, \in)$.

Lemma 16.6: *Suppose X, Y are sets, and θ is an isomorphism between $(X, \in)$ and $(Y, \in)$. Then $X = Y$ and $\theta(x) = x$ for all $x \in X$.*

Proof. By induction on the well-founded relation $\in$. Assume that $\theta(z) = z$ for all $z \in x$ and let $y = \theta(x)$. We have $x \subseteq y$ because if $z \in x$, then $z = \theta(z) \in \theta(x) = y$. We also have $y \subseteq x$: Let $t \in y$. Since $y \in Y$, there is $z \in X$ with $\theta(z) = t$. Since $\theta(z) \in y$, we have $z \in x$, and thus $t = \theta(z) = z \in x$. Hence $x = y$, and this also implies $\theta(x) = x$. □

The lemma, applied to $\pi \circ \rho^{-1}$, yields $S = T$ and $\pi \circ \rho^{-1} = \mathrm{id}$, hence $\pi = \rho$ □

Arithmetizing the satisfaction relation

We can now reformulate membership in L for reals as follows:

$$\alpha \in L \cap \mathbb{N}^{\mathbb{N}} \quad \Leftrightarrow \quad \exists \beta \exists m \, [E_\beta \text{ is an extensional and well-founded relation} \\ \wedge \ (\omega, E_\beta) \models \varphi_{\mathsf{V=L}} \ \wedge \ \pi_\beta(m) = \alpha],$$

where π_β is the Isomorphism of the Mostowski collapse of E_β.

It remains to show that the notion occurring inside the square brackets are definable in second order arithmetic.

Proposition 16.7:

(a) *For any $n \in \mathbb{N}$,*

$$\{(m, \sigma, \gamma) \in \mathbb{N} \times \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : m = \ulcorner \varphi \urcorner \wedge \varphi \textit{ is } \Sigma_n \wedge (\omega, E_\gamma) \models \varphi[\sigma]\}$$

is Σ^0_n.

(b) *If $\alpha \in \mathbb{N}^{\mathbb{N}}$ and E_α is well-founded and extensional, then*

$$\{(m, \gamma) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} : \pi_\alpha(m) = \gamma\}$$

is arithmetical in α.

Proof. See **(author?)** [Kan03]. □

The complexity of well-foundedness

It remains to define the properties "*E_β is extensional*" and "*E_β is well-founded*"

For the first one, notice that

$$E_\beta \text{ is extensional} \quad \Leftrightarrow \quad \forall m, n \, [\forall k (k E_\beta m \leftrightarrow k E_\beta n) \rightarrow m = n].$$

Hence it is arithmetical. On the other hand,

$$E_\beta \text{ is well-founded} \quad \Leftrightarrow \quad \forall \gamma \in \mathbb{N}^{\mathbb{N}} \, \exists n \, \forall m \, [\gamma(n) E_\beta \gamma(m)].$$

Hence being well-founded is a Π^1_1 property. Putting everything together we now have the following.

Theorem 16.8: *The set $L \cap \mathbb{N}^{\mathbb{N}}$ is Σ^1_2.*

In similar way we can show

Theorem 16.9: *The set* $\{(\alpha, \beta) \in (L \cap \mathbb{N}^{\mathbb{N}})^2 : \alpha <_L \beta\}$ *is* Σ^1_2.

If $\mathsf{V} = \mathsf{L}$, then the set is actually Δ^1_2, since then

$$\alpha <_L \beta \quad \Longleftrightarrow \quad \alpha \neq \beta \wedge \neg(\beta <_L \alpha).$$

Finally, since $\mathsf{V} = \mathsf{L}$ implies CH, we can use Proposition 14.4 to show the existence of non-measurable sets under $\mathsf{V} = \mathsf{L}$.

Corollary 16.10: *If* $\mathsf{V} = \mathsf{L}$*, then there exists a* Δ^1_2 *set that is not Lebesgue-measurable and does not have the Baire property.*

Lecture 17: Co-Analytic Sets

In the previous lecture we saw how to translate set theoretic definitions of sets of reals into second order arithmetic. One can ask the converse question – does definability in second order arithmetic imply constructibility? We will see that this is indeed true for Σ^1_2 definable reals. Along the way, we will prove a number of interesting results about Π^1_1 and Σ^1_2 sets.

Normal forms

Analytic sets are projections of closed sets. Closed sets are in $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ are infinite paths through *trees on* $\mathbb{N} \times \mathbb{N}$, i.e. *two-dimensional trees*.

Definition 17.1: A set $T \subseteq \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ is a **two-dimensional tree** if

(i) $(\sigma, \tau) \in T$ implies $|\sigma| = |\tau|$ and

(ii) $(\sigma, \tau) \in T$ implies $(\sigma\restriction_n, \tau\restriction_n) \in T$ for all $n \leq |\sigma|$.

An **infinite branch** of T is a pair $(\alpha, \beta) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ so that

$$\forall n \in \mathbb{N}\, (\alpha\restriction_n, \beta\restriction_n) \in T.$$

As in the one-dimensional case, we use $[T]$ to denote the set of all infinite paths through T. It follows that $A \subseteq \mathbb{N}^\mathbb{N}$ is analytic if and only if there exists a two-dimensional tree T on $\mathbb{N} \times \mathbb{N}$ such that

$$\begin{aligned} \alpha \in A \quad &\Leftrightarrow \quad \exists \beta\, (\alpha, \beta) \in [T] \\ &\Leftrightarrow \quad \exists \beta\, \forall n\, (\alpha\restriction_n, \beta\restriction_n) \in T. \end{aligned}$$

Another way to write this is to put, for given T and $\alpha \in \mathbb{N}^\mathbb{N}$,

$$T(\alpha) = \{\tau : (\alpha\restriction_{|\tau|}, \tau) \in T\}.$$

Then we have, with T witnessing that A is analytic,

$$\alpha \in A \quad \Leftrightarrow \quad T(\alpha) \text{ has an infinite path} \quad \Leftrightarrow \quad T(\alpha) \text{ is not well-founded.}$$

We obtain the following *normal form* for co-analytic sets.

Proposition 17.2: *A set $A \subseteq \mathbb{N}^\mathbb{N}$ is $\boldsymbol{\Pi}^1_1$ if and only if there exists a two-dimensional tree T such that*

$$\alpha \in A \quad \Leftrightarrow \quad T(\alpha) \text{ is well-founded.}$$

If A is (lightface) Π^1_1, then there exists a recursive such T, and the mapping $\alpha \mapsto T(\alpha)$ is computable, as a mapping between reals and trees (which can be coded by reals). This relativizes, i.e. for a $\Pi^1_1(\gamma)$ set, the mapping $\alpha \mapsto T(\alpha)$ is computable in γ. Since computable mappings are continuous, we obtain that the in the above proposition, the mapping $\alpha \mapsto T(\alpha)$ is continuous.

$\boldsymbol{\Pi}^1_1$-complete sets

How does one show that a specific set is *not* Borel? A related question is: Given a definition of a set in second order arithmetic, how can we tell that there is not an easier definition (in the sense that it uses less quantifier changes, no function quantifiers etc.)? The notion of *completeness* for classes in Polish spaces provides a general method to answer such questions.

Definition 17.3: Let X, Y be Polish spaces. We say a set $A \subseteq X$ is **Wadge reducible** to $B \subseteq Y$, written $A \leq_W B$, if there exists a continuous function $f : X \to Y$ such that

$$x \in A \quad \Leftrightarrow \quad f(x) \in B.$$

The important fact about Wadge reducibility is that it preserves classes closed under continuous preimages.

Proposition 17.4: *Let Γ be a family of subsets in various Polish spaces (such as the classes of the Borel or projective hierarchy). If Γ is closed under continuous preimages, then $A \leq_W B$ and $B \in \Gamma$ implies $A \in \Gamma$.*

Proof. If $A \leq_W B$ via f, then $A = f^{-1}(B)$. □

Definition 17.5: A set $A \subseteq X$ is Γ**-complete** is $A \in \Gamma$ and for all $B \in \Gamma$, $B \leq_W A$.

Γ-complete sets can be seen as the most complicated members of Γ. For instance, a $\boldsymbol{\Pi}^1_1$-complete set cannot be Borel, since otherwise every $\boldsymbol{\Pi}^1_1$ set would be Borel, which we have seen is not true. More generally if Γ is any class in the Borel or projective hierarchy, and A is Γ-complete, then A is not in $\neg\Gamma$. For suppose $B \in \Gamma \setminus \neg\Gamma$. Then $B \leq_W A$. If A were also in $\neg\Gamma$, then $B \in \neg\Gamma$, a contradiction.

If $A \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ is $\mathbb{N}^\mathbb{N}$-universal for come class Γ in the Borel or projective hierarchy, then the set

$$\{\langle \alpha, \beta \rangle : (\alpha, \beta) \in A\}$$

is Γ-complete, where $\langle .,. \rangle$ here denotes the pairing function for reals

$$\langle \alpha, \beta \rangle(n) = \begin{cases} \alpha(k) & n = 2k, \\ \beta(k) & n = 2k+1. \end{cases}$$

Since $\langle .,. \rangle$ is continuous, and $B \in \Gamma$ if and only if $B = A_\gamma$ for some $\gamma \in \mathbb{N}^\mathbb{N}$, we have in that case that $B \leq_W A$ via the mapping

$$f(\beta) = \langle \gamma, \beta \rangle.$$

It follows that complete sets exist for all levels of the Borel and projective hierarchy. However, the universal sets they are based on are rather abstract objects. Complete sets are most useful when we can show that a *specific property* implies completeness. We will encounter next an important example for the class of co-analytic sets.

Well-founded relations and well-orderings

In the last lecture we encountered the property of a real coding a well-founded relation: Recall that given $\beta \in \mathbb{N}^\mathbb{N}$, $E_\beta(m,n)$ if and only if $\beta(\langle m,n \rangle) = 0$. Let

$$\mathrm{WF} = \{\beta \in \mathbb{N}^\mathbb{N} \colon E_\beta \text{ is well-founded}\}.$$

Then

$$\beta \in \mathrm{WF} \quad \Leftrightarrow \quad \forall \gamma \in \mathbb{N}^\mathbb{N}\, \exists n\, \forall m\, [\gamma(n) E_\beta \gamma(m)],$$

and hence WF is Π^1_1. A closely related set is

$$\mathrm{WOrd} = \{\beta \in \mathbb{N}^\mathbb{N} \colon E_\beta \text{ is a well-ordering}\}.$$

Then

$$\beta \in \mathrm{WOrd} \quad \Leftrightarrow \quad \beta \in \mathrm{WF} \text{ and } E_\beta \text{ is a linear ordering.}$$

Coding a linear order is easily seen $\mathbf{\Sigma}^1_1$, hence WOrd is Π^1_1, too.

Theorem 17.6: *The sets* WF *and* WOrd *are* $\mathbf{\Pi}^1_1$*-complete.*

Proof. We have seen in Lecture 4 that a tree has an infinite path if and only if the inverse prefix ordering is ill-founded. Trees can be coded as reals, and hence Proposition 17.2 yields immediately that WF is Π^1_1-complete.

For WOrd we use the Kleene-Brouwer ordering (see Lecture 4) and Proposition 4.5. □

The theorem lets us gain further insights in the structure of co-analytic sets. If $\alpha \in \mathbb{N}^{\mathbb{N}}$ codes a well-ordering on $\mathbb{N}$, let

$$\|\alpha\| = \text{ order type of well-ordering coded by } \alpha.$$

It is clear that $\|\alpha\| < \omega_1$. For a fixed ordinal $\xi < \omega_1$, we let

$$\mathrm{WOrd}_\xi = \{\alpha \in \mathrm{WOrd} \colon \|\alpha\| \leq \xi\}.$$

Lemma 17.7: *For any $\xi < \omega_1$, the set WOrd_ξ is Borel.*

Proof. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$. We say $m \in \mathbb{N}$ is in the *domain* of E_α, $m \in \mathrm{dom}(E_\alpha)$, if

$$\exists n\,[mE_\alpha n \,\vee\, nE_\alpha m].$$

It is clear from the definition of E_α that $\mathrm{dom}(E_\alpha)$ is Borel. For $\xi < \omega_1$, let

$$B_\xi = \{(\alpha, n) \colon E_\alpha \mid_{\{m \colon mE_\alpha n\}} \text{ is a well-ordering of order type} \leq \xi\}$$

We show by transfinite induction that every B_ξ is Borel. Suppose B_ζ is Borel for all $\zeta < \xi$. Then, since ξ is countable, $\bigcup_{\zeta<\xi} B_\zeta$ is Borel, too. But

$$(\alpha, n) \in B_\xi \quad \Leftrightarrow \quad \forall m\,[mE_\alpha n \quad \Rightarrow \quad (\alpha, m) \in \bigcup_{\zeta<\xi} B_\zeta],$$

and from this it follows that B_ξ is Borel. Finally, note that

$$\alpha \in \mathrm{WOrd}_\xi \quad \Leftrightarrow \quad \forall n\,[n \in \mathrm{dom}(E_\alpha) \quad \Rightarrow \quad (\alpha, n) \in B_\xi],$$

which implies that WOrd_ξ is Borel. □

Corollary 17.8: *Every $\mathbf{\Pi}^1_1$ set is a union of $\aleph_1$ many Borel sets.*

Proof. Since WOrd is $\mathbf{\Pi}^1_1$-complete, every co-analytic set A is the preimage of WOrd for some continuous function f. We have

$$\mathrm{WOrd} = \bigcup_{\xi<\omega_1} \mathrm{WOrd}_\xi,$$

and hence

$$A = \bigcup_{\xi<\omega_1} f^{-1}(\mathrm{WOrd}_\xi).$$

Since continuous preimages of Borel sets are Borel, the result follows. □

If we work instead with the set

$$C_\xi = \{(\alpha\colon \alpha \in \mathrm{WOrd}_\xi \text{ or } \exists n \in \mathrm{dom}(E_\alpha) \\ [E_\alpha \restriction_{\{m\colon mE_\alpha n\}} \text{ is a well-ordering of order type } \xi]\},$$

then we get that $\mathrm{WOrd} = \bigcap_{\xi<\omega_1} C_\xi$, and hence

Corollary 17.9: *Every $\mathbf{\Pi}^1_1$ set can be obtained as a union or intersection of $\aleph_1$-many Borel sets. Consequently, the same holds for every $\mathbf{\Sigma}^1_1$ set.*

Finally, the previous results allow us to solve the cardinality problem of co-analytic sets at least partially.

Corollary 17.10: *Every $\mathbf{\Pi}^1_1$ set is either countable, of cardinality $\aleph_1$, or of cardinality $2^{\aleph_0}$.*

Lecture 18: $\mathbf{\Sigma}^1_2$ Sets

In this lecture we extend the results of the previous lecture to Σ^1_2 sets.

Tree representations of $\mathbf{\Sigma}^1_2$ sets

Analytic sets are projections of closed sets. Closed sets are in $\mathbb{N}^{\mathbb{N}}$ are infinite paths through trees on ω.

We call a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ *Y-Souslin* if A is the projection $\exists^{Y^{\mathbb{N}}}[T]$ of some $[T]$, where T is a tree on $\mathbb{N} \times Y$, i.e. $A = \exists^{Y^{\mathbb{N}}}[T] = \{\alpha\colon \exists y \in Y^{\mathbb{N}}\, (\alpha, y) \in [T]\}$.

Theorem 18.1 (Shoenfield, 1961)**:** *Every $\mathbf{\Sigma}^1_2$ set is ω_1-Souslin. In particular, if A is Σ^1_2 then there is a tree $T \in L$ on $\mathbb{N} \times \omega_1$ such that $A = \exists^{(\omega_1)^{\mathbb{N}}}[T]$.*

Proof. Assume first A is Π^1_1. There is a recursive tree T on $\mathbb{N} \times \mathbb{N}$ (and hence, in L, since 'being recursive' is definable) such that

$$\alpha \in A \quad \Leftrightarrow \quad T(\alpha) \text{ is well-founded.}$$

Hence, $\alpha \in A$ if and only if there exists an order preserving map $\pi : T(\alpha) \to \omega_1$. We recast this in terms of getting an infinite branch through a tree. Let $\{\sigma_i : i \in \mathbb{N}\}$ be a recursive enumeration of $\mathbb{N}^{<\mathbb{N}}$. We may assume for this enumeration that $|\sigma_i| \leq i$. We define a tree $\widetilde{T}$ on $\mathbb{N} \times \omega_1$ by

$$\widetilde{T} = \{(\sigma, \tau) :\ \forall i, j < |\sigma|\, [\sigma_i \supset \sigma_j \wedge (\sigma\restriction_{|\sigma_i|}, \sigma_i) \in T \ \to\ \tau(i) < \tau(j)]\}.$$

It is easy to see that $\widetilde{T}$ is in L, since it is definable from T and ω_1. Furthermore, if $\alpha \in A$, then the existence of an order-preserving map $\pi : T(\alpha) \to \omega_1$ implies that there is an infinite path (α, η) through $\widetilde{T}$. Conversely, if such a path (α, η) exists, then it is easy to see that there is an order preserving map $\pi : T(\alpha) \to \omega_1$. Hence we have

$$\alpha \in A \longleftrightarrow \exists \eta \in (\omega_1)^{\mathbb{N}}\, (\alpha, \eta) \in [\widetilde{T}] \longleftrightarrow \alpha \in \exists^{(\omega_1)^{\mathbb{N}}}[\widetilde{T}],$$

so A is of the desired form.

Now we extend the representation to Σ^1_2. If A is Σ^1_2, then there is a Π^1_1 set $B \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $A = \exists^{\mathbb{N}^{\mathbb{N}}} B$. Since $B \in \Pi^1_1$, we can employ the tree representation of Π^1_1 to obtain a tree T over $\mathbb{N} \times \mathbb{N} \times \omega_1$ such that $B = \exists^{(\omega_1)^{\mathbb{N}}}[T]$. Now we recast T as a tree T' over $\mathbb{N} \times \omega_1$ such that $\exists^{(\omega_1)^{\mathbb{N}}}[T'] = \exists^{(\omega_1)^{\mathbb{N}}} B$. This is done by using a bijection between $\mathbb{N} \times \omega_1$ and ω_1. This way we can cast the $\mathbb{N} \times \omega_1$ component of T into a single ω_1 component, and thus transform the tree T into a tree T' over $\mathbb{N} \times \omega_1$ such that $\exists^{(\omega_1)^{\mathbb{N}}}[T'] = \exists^{(\omega_1)^{\mathbb{N}}}[B]$. $\square$

$\mathbf{\Sigma}^1_2$ sets as unions of Borel sets

We can use Shoenfield's tree representation to extend Corollary 17.8 to Σ^1_2 sets.

Theorem 18.2 (Sierpiński, 1925)**:** *Every Σ^1_2 set is a union of $\aleph_1$-many Borel sets.*

Sierpinski's original proof used AC. The following proof does not make use of choice.

Proof. Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be Σ^1_2. By Theorem 18.1 there exists a tree T on $\mathbb{N} \times \omega_1$ such that $A = \exists^{(\omega_1)^{\mathbb{N}}}[T]$. For any $\xi < \omega_1$ let

$$T^{\xi} = \{(\sigma, \eta) \in T : \forall i \leq |\eta| \; \eta(i) < \xi\}.$$

Since the cofinality of ω_1 is greater than ω (this can be proved without using AC), every $d : \omega \to \omega_1$ has its range included in some $\xi < \omega_1$. Thus we have

$$A = \bigcup_{\xi < \omega_1} \exists^{(\omega_1)^{\mathbb{N}}}[T^{\xi}].$$

For all $\xi < \omega_1$, the set $\exists^{(\omega_1)^{\mathbb{N}}}[T^{\xi}]$ is Σ^1_1, because the tree T^{ξ} is a tree on a product of countable sets and hence is isomorphic to a tree on $\mathbb{N} \times \mathbb{N}$. By Corollary 17.9, each Σ^1_1 set is the union of $\aleph_1$ many Borel sets, from which the result follows. $\square$

Again, an immediate consequence of this theorem is (using the perfect set property of Borel sets):

Corollary 18.3: *Every $\mathbf{\Sigma}^1_2$ set has cardinality at most $\aleph_1$ or has a perfect subset and hence cardinality $2^{\aleph_0}$.*

Absoluteness of Σ^1_2 relations

Shoenfield used the tree representation of $\mathbf{\Sigma}^1_2$ sets to establish an important absoluteness result for Σ^1_2 sets of reals.

Suppose $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ^1_2. Then, by the Kleene Normal Form there exists a bounded formula $\varphi(\alpha, \beta_0, \beta_1, m)$ such that

$$\alpha \in A \quad \Leftrightarrow \quad \exists \beta_0 \, \forall \beta_1 \, \exists m \; \varphi(\alpha, \beta_0, \beta_1, m).$$

Let M be in inner model of ZF, i.e. M is transitive and contains all ordinals. Since arithmetical formulas can be interpreted in ZF, M contains all recursive predicates over $\mathbb{N}$. In particular, since the truth of the bounded formula φ depends only on finite initial segments of α, β_0, β_1, it defines a recursive predicate $R_\varphi(\alpha, \beta_0, \beta_1, m) = R_\varphi(\sigma, \tau_0, \tau_1, m)$, which in turns defines a subset of $\mathbb{N}^4$ that is contained in M. Hence we can define the *relativization* of A to M as

$$A^M(\alpha) \quad \Leftrightarrow \quad \exists \beta_0 \in M \, \forall \beta_1 \in M \, \exists m \, R(\alpha, \beta_0, \beta_1, m).$$

We say that A is **absolute for** M if for any $\alpha \in M$,

$$A^M(\alpha) \quad \Leftrightarrow \quad A(\alpha).$$

Absoluteness itself can be extended and relativized in a straightforward manner to predicates analytical in some $\gamma \in \mathbb{N}^{\mathbb{N}} \cap M$.

Theorem 18.4 (Shoenfield Absoluteness)**:** *Every $\Sigma^1_2(\gamma)$ predicate and every $\Pi^1_2(\gamma)$ predicate is absolute for all inner models M of ZFC such that $\gamma \in M$. In particular, all Σ^1_2 and Π^1_2 relations are absolute for L.*

Proof. We show the theorem for Σ^1_2 predicates. For the relativized version, one uses the *relative constructible universe* $L[\gamma]$, see **(author?)** [Jec03] or **(author?)** [Kan03].

Let A be a Σ^1_2 relation. For simplicity, we assume that A is unary. Fix a tree representation of A as a projection of a Π^1_1 set. So, let T be a recursive tree on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$\alpha \in A \quad \Leftrightarrow \quad \exists \beta \; T(\alpha, \beta) \text{ is well-founded.}$$

Note that T is in M.

Now assume $\alpha \in M$ and $\alpha \in A^M$. So there is a $\beta \in M$ such that $T(\alpha, \beta)$ is well-founded in M. This is equivalent to the fact that in M there exists an order preserving mapping $\pi : T(\alpha, \beta) \to \mathrm{Ord}^M$. Since M is an inner model and T is the same in V and M, such a mapping exists also in V. Hence $T(\alpha, \beta)$ is well-founded in V and thus $\alpha \in A$.

For the converse assume that $\alpha \in A \cap M$. Now we use the tree representation of A given by Theorem 18.1. Let $U \in L \subseteq M$ be a tree on $\mathbb{N} \times \omega_1$ such that $A = \exists^{(\omega_1)^{\mathbb{N}}} U$. This means that for any $\alpha \in \mathbb{N}^{\mathbb{N}}$,

$$\alpha \in A \quad \Longleftrightarrow \quad U(\alpha) \text{ is not well-founded.}$$

So $\alpha \in A \cap M$ implies that there exists no order preserving map $U(\alpha) \to \omega_1$. But then such a map cannot exist in M either. So, $U(\alpha)$ is a tree in M which is ill-founded in the sense of M. Thus, by Shoenfield's Representation Theorem relativized to M, $\alpha \in A^M$.

Absoluteness for Π^1_2 follows by employing the same reasoning, using that the complement is Σ^1_2. $\square$

By analyzing the proof one sees that it actually suffices that M is a transitive $\in$-model of a certain finite collection of axioms ZF such that $\omega_1 \subseteq M$.

The result is the best possible with respect to the analytical hierarchy, since the statement

$$\exists \alpha\, [\alpha \notin L]$$

is Σ^1_3, but cannot be absolute for $M = L$.

Shoenfield's Absoluteness Theorem also holds for sentences rather than formulae, with a similar proof. This means a Σ^1_2 statement is true in L if and only if it holds in V. This has an important consequence regarding the significance of principles like CH for analysis. Many results of classical analysis are Σ^1_2 statements. The Absoluteness Theorem says that if they can be established under $\mathsf{V} = \mathsf{L}$ (and hence in a world where CH holds), they can be established in ZF alone.

Another consequence concerns the complexity of reals defined by analytical relations.

Corollary 18.5: *If $X \subseteq \omega$ is Σ^1_2, then $X \in L$. In particular, every Σ^1_2 real (and hence every Π^1_2 real) is in L.*

Proof. Let X be Σ^1_2 via some formula φ. Since $\omega \in L$, and since L is an inner model of ZF, it satisfies the axiom of separation (relativized to L) for φ. So the set $X^L = \{a \in \omega : \varphi^L(a)\}$ is in L. It is clear that the representation and absoluteness results also hold for subsets of ω. (Change the notation to include subsets of ω.) Absoluteness for φ implies that $X^L \cap L = X \cap L$, but since $X \subseteq \omega$, we have $X = X \cap L$ and $X^L \cap L = X^L$, and hence $X \in L$. $\square$

We cannot extend this to Σ^1_2 sets of reals. In the proof of the Corollary, it is crucial that ω, the set over which we apply separation, is in L. This is not longer the case for sets of reals. For example, the set of all reals is clearly Σ^1_2, but unless $\mathsf{V} = \mathsf{L}$, it is not contained in L.

Lecture 19: Recursive Ordinals and Ordinal Notations

We have seen that the property "*α codes a well-ordering of* $\mathbb{N}$" is important for the study of co-analytic sets. If A is $\boldsymbol{\Pi}^1_1$, then there exists a tree T on $\mathbb{N}\times\mathbb{N}$ such that

$$\alpha \in A \quad \Leftrightarrow \quad T(\alpha) \text{ is well-founded.}$$

If $T(\alpha)$ is well-founded, then the Kleene-Brouwer ordering restricted to T is a well-ordering. Since $T(\alpha)$ is a tree on $\mathbb{N}$, it constitutes an ordering on $\mathbb{N}$, using a standard bijection between strings and natural numbers.

If A is moreover Π^1_1, then there is a *recursive* such tree and the tree $T(\alpha)$ is recursive in α. If α is recursive and $\alpha \in A$, then $T(\alpha)$ encodes a *recursive well-ordering*.

In general, we say an ordinal $\xi < \omega_1$ is **recursive** if there exists a recursive $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that

$$E_\alpha = \{(m,n) \in \mathbb{N}\times\mathbb{N} \colon \alpha(\langle m,n\rangle) = 0\}$$

is a well-ordering of order type ξ.

Proposition 19.1: *The recursive ordinals form a countable initial segment of the class* Ord *of all ordinals.*

Proof. Suppose ξ is a recursive ordinal. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$ be recursive so that the order type of E_α is ξ. Let $\eta < \xi$. Since $\eta \in \xi$, there exists n such that E_α restricted to

$$\{m \colon mE_\alpha n\}$$

has order type η. Hence η is recursive via the relation

$$\{(k,m) \colon k, mE_\alpha n \ \&\ kE_\alpha m\}.$$

Thus the set of all recursive ordinals forms an initial segment of Ord. The initial segment is countable since there are only countably many recursive relations. □

There must exist a least non-recursive ordinal (which is countable). This ordinal is called ω_1^{CK}. The CK stands for *Church-Kleene*.

Ordinal Notations

The definition of a recursive ordinal is rather from *outside*. As we will see later, deciding whether a recursive relation defines a well-ordering is quite difficult.

To get a better handle on recursive ordinals, we will construct them from *inside*. The idea is that if we have constructed ξ, then we also know how to construct $\xi + 1$. If we have a sequence of ordinals (ξ_n) previously constructed, we can also construct their limit, provided the sequence itself is constructive.

To make this precise, we introduce *ordinal notations*. A notation system for ordinals assigns ordinals to natural numbers in a way that reflects how each ordinal is built up from its predecessors. Our exposition in this part follows **(author?)** [Rog87].

Definition 19.2 (Kleene)**:** A **system of notation** S is a mapping ν_S from a set $D_S \subseteq \mathbb{N}$ onto an initial segment of Ord such that

(a) there exists a partial recursive function k_S such that

$$\begin{aligned} \nu_S(x) = 0 \quad &\Rightarrow \quad k_S(x) = 0 \\ \nu_S(x) \text{ successor} \quad &\Rightarrow \quad k_S(x) = 1 \\ \nu_S(x) \text{ limit} \quad &\Rightarrow \quad k_S(x) = 2; \end{aligned}$$

(b) there exists a partial recursive function p_S such that

$$\nu_S(x) \text{ successor} \quad \Rightarrow \quad [p_s(x)\downarrow \;\&\; \nu_S(x) = \nu_S(p_S(x)) + 1];$$

(c) there exists a partial recursive function q_S such that

$$\begin{aligned} \nu_S(x) \text{ limit} \quad \Rightarrow \quad [&q_S(x)\downarrow \;\&\; \varphi_{q_S(x)} \text{ total } \;\&\; (\nu_S(\varphi_{q_S(x)}(n)))_{n\in\mathbb{N}} \\ &\text{is an increasing sequence with limit } \nu_S(x)]. \end{aligned}$$

Given an ordinal notation $x \in D_S$ we can hence decide whether x codes a successor or a limit ordinal (or 0), and we can determine a notation for the predecessor of x (if x is a successor), or an index for a sequence of notations of ordinals converging to the ordinal denoted by x.

Note that the conditions ensure that the ordinals with a notation in S actually form an initial segment of Ord. This follows by induction. Note further that we do not require ν_S to be one-one. An ordinal may receive multiple notations.

Definition 19.3: An ordinal ξ is **constructive** if there exists a system of notation that assigns at least one notation to ξ.

Of course there are many different systems of notation. We would like to have one that encompasses *all constructive ordinals*, that is, a universal system.

Definition 19.4: A system of notation S is **universal** if for any system S' there is a partial recursive function φ such that $\varphi(D_{S'}) \subseteq D_S$ and

$$x \in D_{S'} \quad \Rightarrow \quad \nu_{S'}(x) \leq \nu_S(\varphi(x))$$

Since systems are closed downwards, this means that S assigns a notation to $\nu_{S'}(x)$, too.

It is a remarkable result due to Kleene that universal systems exist.

The system S_1

We define the system S_1 recursively.

- 0 receives notation 1.
- If all ordinals $< \xi$ have received their notations then
  (a) if $\xi = \eta + 1$, ξ receives the notations $\{2^x : \ x \text{ is a notation for } \eta\}$,
  (b) if ξ is limit, ξ receives the notation $3 \cdot 5^y$ for each y such that for all n, $\varphi_y(n)$ is a notation and the ordinals denoted by the $\varphi_y(n)$ form a sequence with limit ξ.

The functions $k_{S_1}, p_{S_1}, q_{S_1}$ are easily defined as

$$k_{S_1}(x) = \begin{cases} 0 & x = 1 \\ 1 & x = 2^y \\ 2 & x = 3 \cdot 5^y \\ \uparrow & \text{otherwise} \end{cases}$$

and

$$p_{S_1}(2^x) = x \qquad q_{S_1}(3 \cdot 5^y) = y,$$

where $p_{S_1}(z)$ and $q_{S_1}(z)$ are undefined in all other cases.

One can show that S_1 is universal (see **(author?)** [Rog87]). We will impose additionally an ordering on the ordinal notations of S_1. This will be useful later. The result is the system $\mathcal{O}$.

The system $\mathcal{O}$

We define simultaneously a system of notations and an ordering $<_{\mathcal{O}}$ on notations.

- 0 receives notation 1.
- Suppose all ordinals $< \xi$ have received their notations, and assume that $<_{\mathcal{O}}$ has been defined on these notations.
  (a) if $\xi = \eta + 1$, ξ receives the notations $\{2^x : x \text{ is a notation for } \eta\}$, and set $z <_{\mathcal{O}} 2^x$ for $z = x$ or $z <_{\mathcal{O}} x$.
  (b) if ξ is limit, ξ receives the notation $3 \cdot 5^y$ for each y such that for all n, $\varphi_y(n)$ is a notation and the ordinals denoted by the $\varphi_y(n)$ form a sequence with limit ξ, and for all $i < j$, $\varphi_y(i) <_{\mathcal{O}} \varphi_y(j)$. Furthermore, for each such y, set $z <_{\mathcal{O}} 3 \cdot 5^y$ for any z with $z <_{\mathcal{O}} \varphi_y(n)$ for some n.

The function $k_{\mathcal{O}}, p_{\mathcal{O}}, q_{\mathcal{O}}$ are identical with $k_{S_1}, p_{S_1}, q_{S_1}$.

We will denote $D_{\mathcal{O}}$ by $\mathcal{O}$, too. Instead of $\nu_{\mathcal{O}}(x)$ we write $|x|_{\mathcal{O}}$. $<_{\mathcal{O}}$ is a partial ordering on $\mathcal{O}$. Its transitivity follows from the definition of $<_{\mathcal{O}}$. An effective limit of ordinals with notations in $\mathcal{O}$ can (and does) have many possible indices. This makes the ordering non-linear. This is reflected in the following diagram of the initial structure of $<_{\mathcal{O}}$.

$$1 <_{\mathcal{O}} 2 <_{\mathcal{O}} 2^2 <_{\mathcal{O}} \dots \quad \left\{ \begin{array}{l} 3 \cdot 5^{y_1} <_O 2^{3 \cdot 5^{y_1}} <_{\mathcal{O}} 2^{2^{3 \cdot 5^{y_1}}} <_{\mathcal{O}} \dots \\ 3 \cdot 5^{y_2} <_O 2^{3 \cdot 5^{y_2}} <_{\mathcal{O}} 2^{2^{3 \cdot 5^{y_2}}} <_{\mathcal{O}} \dots \\ \vdots \end{array} \right.$$

$3 \cdot 5^{y_1}$ and $3 \cdot 5^{y_2}$ are two of the infinitely many notations for ω. Any index y of the recursive function that maps x to x-many iterations of $n \mapsto 2^n$ constitutes a notation $3 \cdot 5^y$ of ω. For this reason $|x|_{\mathcal{O}} < |y|_{\mathcal{O}}$ does not necessarily imply $x <_{\mathcal{O}} y$.

However, it is easy to see that $x <_{\mathcal{O}} y$ implies $|x|_{\mathcal{O}} < |y|_{\mathcal{O}}$. Since an infinite descending sequence in $<_{\mathcal{O}}$ would induce an infinite descending sequence in Ord, we have

Proposition 19.5: *The relation $<_{\mathcal{O}}$ is well-founded.*

This allows us to prove facts about $<_{\mathcal{O}}$ via induction along a well-founded relation.

Proposition 19.6: *Let $y \in \mathcal{O}$. Then*

(a) the restriction of $<_{\mathcal{O}}$ to $\{x : x <_{\mathcal{O}} y\}$ is linear;

(b) the restriction of $\mathcal{O}$ to $<_{\mathcal{O}}$ to $\{x : x <_{\mathcal{O}} y\}$ is one-one.

Proof. (a) We proceed by induction along $<_{\mathcal{O}}$. Suppose $x_1, x_2 <_{\mathcal{O}} y$. If $y = 2^z$, then $z <_{\mathcal{O}} y$ and by definition of $<_{\mathcal{O}}$ if $v <_{\mathcal{O}} 2^z$ then $v \leq_{\mathcal{O}} z$. Hence $x_1, x_2 \leq_{\mathcal{O}} z$ and we can apply the induction hypothesis. If $y = 3 \cdot 5^z$, then by definition of $<_{\mathcal{O}}$ there exist n_1, n_2 such that $x_1 <_{\mathcal{O}} \varphi_z(n_1)$ and $x_2 <_{\mathcal{O}} \varphi_z(n_2)$. Wlog $n_1 < n_2$. Then, by the condition for y to be a notation we have $\varphi_z(n_1) <_{\mathcal{O}} \varphi_z(n_2)$, and hence $x_1, x_2 <_{\mathcal{O}} \varphi_z(n_2)$, and we can apply the induction hypothesis.

(b) This is an easy induction – each step in the definition of $\mathcal{O}, <_{\mathcal{O}}$ defines a notation for an ordinal larger than all ordinals having received a notation before. $\square$

We can also show

Proposition 19.7: *The restriction of $<_{\mathcal{O}}$ to $\{y : y <_{\mathcal{O}} x\}$ is uniformly r.e., i.e. there exists a recursive function f such that for all x, if $x \in \mathcal{O}$ then $W_{f(x)} = \{y : y <_{\mathcal{O}} x\}$.*

We defer the proof for a while to discuss the use of the Recursion Theorem.

Effective Transfinite Recursion

The Recursion Theorem plays an essential role in computations with ordinal notations. To see why, consider the following problem. We would like to introduce a (partial) recursive function $+_{\mathcal{O}}$ that mirrors the addition of ordinals on the notational side. More specifically we would like a function $+_{\mathcal{O}}$ such that for all $x, y \in \mathcal{O}$,

(a) $x +_{\mathcal{O}} y \in \mathcal{O}$,

(b) $|x +_{\mathcal{O}} y|_{\mathcal{O}} = |x|_{\mathcal{O}} + |y|_{\mathcal{O}}$, and

(c) $y \neq 1$ implies $x <_{\mathcal{O}} x +_{\mathcal{O}} y$.

The obvious way to define such a function $+_{\mathcal{O}}$ is by recursion. Suppose we fix x and try to define $x +_{\mathcal{O}} y$. It is clear that $x +_{\mathcal{O}} 1 = x$. If $y = 2^z$, we define

$x +_{\mathcal{O}} y = 2^{x +_{\mathcal{O}} z}$. Now suppose $y = 3 \cdot 5^z$. To match the definition of ordinal addition, we have to put $x +_{\mathcal{O}} y$ to be the "*limit*" of the notations $x +_{\mathcal{O}} \varphi_z(n)$. In other words, we have to set $x +_{\mathcal{O}} y = 3 \cdot 5^e$, where e is an index of the computable mapping $n \mapsto x +_{\mathcal{O}} \varphi_z(n)$. Hence to determine e, we need an index for the very function we are trying to build!

This is where the Recursion Theorem is indispensable. It ensures us that we know such an index "beforehand". The following theorem captures this possibility of **effective transfinite recursion**.

Theorem 19.8 (Effective transfinite recursion)**:** *Let R be a well-founded relation defined on a subset of $\mathbb{N}$. Suppose $F : \mathbb{N} \to \mathbb{N}$ is a (total) recursive function. Suppose further that for all $e \in \mathbb{N}$ and $x \in \mathrm{dom}(R)$,*

$$\forall y \, R \, x \ \varphi_e(y) \downarrow \quad \Rightarrow \quad \varphi_{F(e)}(x) \downarrow .$$

Then there exists a $c \in \mathbb{N}$ such that

$$\forall x \in \mathrm{dom}(R) \ \varphi_c(x) \downarrow \quad \textit{and} \quad \varphi_c = \varphi_{F(c)}.$$

The idea is that if we have efficiently constructed a function (i.e. an index e) below x, and given this index we effectively compute an extension to x (via $\varphi_{F(e)}$), then we actually succeeded in effectively constructing a function defined on all of $\mathrm{dom}(R)$. This is precisely the situation we are facing in the definition of $x +_{\mathcal{O}} y$.

Proof. By the Recursion Theorem there exists a c such that $\varphi_c = \varphi_{F(c)}$. If $\varphi_c(x)$ were undefined for some $x \in \mathrm{dom}(R)$, then, since R is well-founded, there must exist an R-minimal such x. This implies that $\varphi_c(y)$ is defined for all $y \, R \, x$, and hence by assumption, $\varphi_{F(c)}(x) \downarrow$. Since $\varphi_{F(c)} = \varphi_c$, this is a contradiction. □

Armed with effective transfinite recursion, we can give a formal construction of the function $+_{\mathcal{O}}$. Using the S-m-n Theorem, we can fix an injective function $h : \mathbb{N}^3 \to \mathbb{N}$ such that

$$\varphi_{h(e,x,d)}(y) = \varphi_e(x, \varphi_d(y)) \quad \text{for all } e, x, d, y \in \mathbb{N}.$$

Let F be a recursive function such that for all e,

$$\varphi_{F(e)}(x, y) \simeq \begin{cases} x & \text{if } y = 1, \\ 2^{\varphi_e(x,z)} & \text{if } y = 2^z, \\ 3 \cdot 5^{h(e,x,z)} & \text{if } y = 3 \cdot 5^z, \\ 7 & \text{otherwise.} \end{cases}$$

Let c be a fixed point of F and put $x +_{\mathcal{O}} y = \varphi_c(x, y)$. It is straightforward to verify that this definition of $+_{\mathcal{O}}$ has the desired properties. Note that for the definition to work, it is essential that we can distinguish effectively between codes for 0, successor, and limit ordinals (the recursiveness of $k_{\mathcal{O}}$).

Maybe surprisingly, $+_{\mathcal{O}}$ turns out to be total. Suppose $\langle x, y\rangle$ is minimal (with respect to the usual ordering of $\mathbb{N}$) so that $x +_{\mathcal{O}} y$ is undefined. Since h is total, the only way for $x +_{\mathcal{O}} y$ to be undefined is for y to be of the form 2^z. But this means $\varphi_c(x,z)\uparrow$, and hence $x +_{\mathcal{O}} z$ is undefined for some lesser pair $\langle x, z\rangle$, under the standard pair coding function.

We can use $+_{\mathcal{O}}$ to prove the universality of $\mathcal{O}$.

Proposition 19.9: *$\mathcal{O}$ is a universal system of notation.*

Proof. Let S be a system of notation. Again, we use effective transfinite recursion. Let h be a recursive function such that

$$\begin{aligned} \varphi_{h(e)}(0) &= \varphi_e(\varphi_{q_S(x)}(0)), \\ \varphi_{h(e)}(x+1) &= \varphi_{h(e)}(x) +_{\mathcal{O}} +\varphi_e(\varphi_{q_S(x)}(x+1)). \end{aligned}$$

Recall that $+_{\mathcal{O}}$ is total. Define a recursive function F such that

$$\varphi_{F(e)}(x) \simeq \begin{cases} 1 & \text{if } k_S(x) = 0, \\ 2^{\varphi_e(p_S(x))} & \text{if } k_S(x) = 1, \\ 3 \cdot 5^{h(e)} & \text{if } y = 3 \cdot 5^z, \\ \uparrow & \text{otherwise.} \end{cases}$$

The Recursion Theorem yields a fixed point $\varphi_{F(c)} = \varphi_c$. Then φ_c is the desired reduction. Suppose not. Then there exists an least ξ such that $\nu_S(x) = \xi$ for some x, but $|\varphi_c(x)|_{\mathcal{O}} < \nu_S(x)$. If $\xi = \eta + 1$, then $\varphi_c(x) = 2^{\varphi_c(p_S(x))}$, and $|\varphi_c(p_S(x))|_{\mathcal{O}} < |\varphi_c(x)|_{\mathcal{O}} \leq \eta = \nu_S(p_S(x))$, contradicting the fact that ξ was chosen minimal. The case that ξ is limit is similar. $\square$

Finally, we give a proof of Proposition 19.7.

Proof of Proposition 19.7. We follow **(author?)** [Sac90]. We claim that there exists a recursive function f such that

$$W_{f(1)} = \emptyset,$$
$$W_{f(2^x)} = W_{f(x)} \cup \{x\},$$
$$W_{f(3\cdot 5^x)} = \bigcup\{W_{f(\varphi_x(n))} \colon \varphi_x(n)\downarrow\}.$$

It follows by induction along $<_{\mathcal{O}}$ that such f satisfies the assertion of the theorem. Choose an index e_0 and recursive functions h_0, h_1 such that

$$W_{e_0} = \emptyset,$$
$$W_{h_0(e,x)} = W_{\varphi_e(x)} \cup \{x\},$$
$$W_{h_1(e,x)} = \bigcup\{W_{\varphi_e(\varphi_x(n))} \colon n \in \mathbb{N}\}.$$

Here $W_{\varphi_e(x)} = \emptyset$ if $\varphi_e(x)\uparrow$; similarly for $W_{\varphi_e(\varphi_x(n))}$. There exists a recursive function F such that

$$\varphi_{F(e)}(x) \simeq \begin{cases} e_0 & \text{if } x = 1, \\ h_0(e,z) & \text{if } x = 2^z, \\ h_1(e,z) & \text{if } x = 3\cdot 5^z, \\ 0 & \text{otherwise.} \end{cases}$$

Let c be a fixed point of F and define $f(x) = \varphi_c(x)$. Note that f is total because h_0, h_1 are. □

The last two result puts us in a position to prove

Theorem 19.10: *A constructive ordinal is recursive.*

Proof. Let α be a constructive ordinal. Since $\mathcal{O}$ is universal, it assigns α a notation, say x. By Proposition 19.7 the set set $\mathcal{O}_x = \{y \colon y <_{\mathcal{O}} x\}$ is r.e. A slight variation of the proof of Proposition 19.7 yields that the set $\mathcal{O}_x^< = \{\langle y, z\rangle \colon y <_{\mathcal{O}} z <_{\mathcal{O}} x\}$ is r.e., too. We may assume that $\mathcal{O}_x$ is infinite. By Proposition 19.6 $\mathcal{O}_x^<$ is well-founded and linear, hence a well-ordering. An easy induction shows that the order type of $\mathcal{O}_x^<$ is $|x|_{\mathcal{O}} = \alpha$. Let f be recursive, one-one such that $\operatorname{ran}(f) = \mathcal{O}_x$. Put

$$m R n \iff \langle f(m), f(n)\rangle \in \mathcal{O}_x^<.$$

Since $\mathcal{O}_x = \operatorname{ran}(f)$ is the domain of $\mathcal{O}_x^<$ and $\mathcal{O}_x^<$ is r.e., it follows that R is a recursive well-ordering of order type α. □

We will see in the next lecture that the converse is also true. This will be a consequence of the **completeness properties of** $\mathcal{O}$, which we study next.

Lecture 20: Π^1_1 Sets of Natural Numbers

In this lecture we consider Π^1_1 sets of natural numbers. They are defined just like their counterparts in $\mathbb{N}^{\mathbb{N}}$. Using the Kleene Normal Form, a set $X \subseteq \mathbb{N}$ is Π^1_1 if there exists a bounded formula $\varphi(x, y, \beta,)$ such that

$$x \in X \quad \Leftrightarrow \quad \forall \beta \exists y\, \varphi(x, y, \beta).$$

One can show that equivalently, there exists a recursive relation $R(x, y, \beta)$ such that

$$x \in X \quad \Leftrightarrow \quad \forall \beta \exists y\, R(x, y, \beta).$$

Σ^1_1 sets are given analogously.

Recursive relations are those that are Σ^0_1 and Π^0_1 at the same time, i.e. that are Δ^0_1. There are recursive relations that are *not* definable by bounded formulas. Hence the above equivalence requires a little bit of work, for which we refer to **(author?)** [Kan03].

On the other hand, one can show that a relation $R(x, y, \beta)$ is recursive if and only there exists an e such that for all x, y, β, $\Phi^\beta_e(x, y)\downarrow$ and

$$R(x, y, \beta) \quad \Leftrightarrow \quad \Phi^\beta_e(x, y) = 0.$$

The truth of the right hand side depends only on a finite initial segment of β (the *use principle*). This is reflected by the **Kleene T-predicate**. This is a recursive predicate T such that, for some recursive function U,

$$\Phi^\beta_e(x, y) \simeq 0 \quad \Leftrightarrow \quad \exists s\, [T(e, x, y, s, \beta\restriction_s)\, \&\, U(s) = 0].$$

Hence we have (using quantifier contraction) that $X \subseteq \mathbb{N}$ is Π^1_1 if and only if there exists a recursive predicate R^* such that

$$x \in X \quad \Leftrightarrow \quad \forall \beta \exists y\, R^*(x, y, \beta\restriction_y).$$

This allows us to derive a tree representation similar to the case of Baire space. Namely, let

$$\sigma \in S(x) \quad \Leftrightarrow \quad \forall i < |\sigma|\, \neg R^*(x, \sigma\restriction_i, i).$$

Then $S(x)$ is a recursive tree for each x, and

$$x \in X \quad \Leftrightarrow \quad S(x) \text{ is well-founded.}$$

Kleene's $\mathcal{O}$ and well-founded relations

The above normal form reduces deciding membership in a Π^1_1 set to deciding whether a recursive predicate is well-founded. We will now show that $\mathcal{O}$ can decide the latter question in a uniform way.

Let the e-th r.e. relation $R_e(x, y)$ be given by

$$x\, R_e\, y \quad \Leftrightarrow \quad R_e(x, y) \quad \Leftrightarrow \quad \varphi_e(x, y)\downarrow.$$

As before, the domain of R_e, $\mathrm{dom}(R_e)$ is given as

$$\mathrm{dom}(R_e) = \{x : \exists y\, R_e(x, y) \vee R_e(y, x)\}.$$

Note that $\mathrm{dom}(R_e)$ is r.e., too.

Let $\mathrm{WF}_{\mathbb{N}} = \{e : R_e \text{ is well-founded}\}$. We want to show that $\mathrm{WF}_{\mathbb{N}}$ reduces to $\mathcal{O}$. To this end we define, uniformly, an effective order-preserving mapping f from $\mathrm{dom}(R_e)$ into $\mathcal{O}$. We do this by effective transfinite recursion. Let $h(e, n)$ be a recursive function such that

$$R_{h(e,n)}(x, y) \quad \Leftrightarrow \quad R_e(x, n) \,\&\, R_e(y, n) \,\&\, R_e(x, y),$$

where $R_{h(e,n)}$ is empty if $n \notin \mathrm{dom}(R_e)$. $R_{h(e,n)}$ is the initial segment of R_e below n. Clearly the $R_{h(e,n)}$ are uniformly enumerable. Since we can enumerate R_e by enumerating the $R_{h(e,n)}$, the idea is to define f on the $R_{h(e,n)}$, and then extend f to R_e by transfinite recursion.

The images of the $R_{h(e,n)}$ will be r.e., too. Moreover, we can enumerate these images uniformly, obtaining an r.e. subset of $\mathcal{O}$. To extend our mapping f to R_e, we need an effective way to find, given an r.e. subset W of $\mathcal{O}$, an element of $\mathcal{O}$ "on top" of W.

The next lemma shows that we can do this in a uniform way.

Lemma 20.1: *There exists a recursive function g such that*

(a) $g(e) \in O$ if and only if $W_e \subseteq \mathcal{O}$,

(b) If $g(e) \in \mathcal{O}$, then $|x|_{\mathcal{O}} < |g(e)|_{\mathcal{O}}$ for all $x \in W_e$.

Proof. We use recursion along the $+_{\mathcal{O}}$ function, summing the elements of W_e. To ground the recursion, we first add 1 to W_e: Let $r(e)$ be recursive such that

$\operatorname{ran}(\varphi_{r(e)}) = W_e \cup \{1\}$ and $\varphi_{r(e)}(0) = 1$. Now define a recursive function s such that

$$\varphi_{s(e)}(0) = \varphi_{r(e)}(0) = 1,$$
$$\varphi_{s(e)}(n+1) = \varphi_{s(e)}(n) +_{\mathcal{O}} 2^{\varphi_{r(e)}(n+1)}.$$

We put $g(e) = 3 \cdot 5^{s(e)}$.

We verify (a). Suppose $g(e) \in \mathcal{O}$. Then $\varphi_{s(e)}(n) \in \mathcal{O}$ for all n. It is not hard to show that $x +_{\mathcal{O}} y \in \mathcal{O}$ if and only if $x, y \in \mathcal{O}$. Therefore, $2^{\varphi_{r(e)}(n)} \in \mathcal{O}$ and hence $\varphi_{r(e)}(n) \in \mathcal{O}$ for all n. Now assume $W_e \subseteq \mathcal{O}$. It follows that for each n, $\varphi_{r(e)}(n) \in \mathcal{O}$. By the properties of $+_{\mathcal{O}}$, $\varphi_{s(e)}(n)$ for all n and $\varphi_{s(e)}(n) <_{\mathcal{O}} \varphi_{s(e)}(n+1)$. Hence $g(e) \in \mathcal{O}$.

For (b), suppose $g(e) \in \mathcal{O}$. By definition of g we have $g(e) >_{\mathcal{O}} 1$, so let $1 \neq a \in W_e$. We can choose $n > 0$ such that $\varphi_{r(e)}(n) = a$. By definition of g, $g(e) >_{\mathcal{O}} \varphi_{s(e)}(n)$ for all n. We have

$$\varphi_{s(e)}(n) = \varphi_{s(e)}(n-1) +_{\mathcal{O}} 2^a.$$

Therefore $2^a \leq_{\mathcal{O}} g(e)$ and thus $a <_{\mathcal{O}} g(e)$. $\square$

We have to deal with the possibility that $\operatorname{dom}(R_e)$ is empty, in wich case our recursion would get stuck at the very beginning and not return a value. We prevent this by dealing with this case explicitly. Let t be recursive such that

$$W_{t(b,e)} = \begin{cases} \emptyset & \text{if } R_e = \emptyset, \\ \{\varphi_b(h(e,n)) \colon n \in \mathbb{N}\} & \text{otherwise.} \end{cases}$$

Think of b as an index for f. We choose a recursive function k such that

$$\varphi_{k(b)}(e) \simeq g(t(e,b)).$$

Let c be a fixed point of k. We put

$$f(e) = \varphi_c(e),$$
$$t(e) = t(c,e).$$

Then

$$W_{t(e)} = \begin{cases} \emptyset & \text{if } R_e = \emptyset, \\ \{f(h(e,n)) \colon n \in \mathbb{N}\} & \text{otherwise,} \end{cases}$$

and hence $f(e) = g(t(e))$.

Suppose R_e is well-founded. If $\mathrm{dom}(R_e) = \emptyset$, then $W_{t(e)} = \emptyset \subseteq \mathcal{O}$ and $f(e) \in O$ by the Lemma. If $R_e \neq \emptyset$, then it follows by induction that $R_{h(e,n)} \subseteq \mathcal{O}$ for all n, and by definition of f, $f(e) \in \mathcal{O}$, using Lemma 20.1.

If, on the other hand, $f(e) \in \mathcal{O}$, then, by Lemma 20.1, $|f(e)|_{\mathcal{O}} > |a|_{\mathcal{O}}$ for all $a \in W_{t(e)}$. But the elements of $W_{t(e)}$ are precisely the numbers $f(h(e,n))$. By transfinite induction on $<_{\mathcal{O}}$, this means that each $R_{h(e,n)}$ is well-founded. Hence R_e is well-founded.

Summing up, we have shown

Theorem 20.2: $\mathrm{WF}_{\mathbb{N}}$ *many-one reduces to* $\mathcal{O}$.

The proof of Theorem 20.2 also yields that $|f(e)|_{\mathcal{O}}$ bounds the rank $\rho(R_e)$ of R_e, provided R_e is well-founded. The rank of R_e in this case is simply the rank of the corresponding tree.

Corollary 20.3: *There exists a recursive function f such that if R_e is well-founded, then $\rho(R_e) \leq |f(e)|_{\mathcal{O}}$.*

Theorem 20.2 also lets us show that every recursive ordinal is constructive.

Proposition 20.4: *Every recursive ordinal is constructive.*

Proof. Suppose ξ is recursive. Let R be a recursive well-ordering of $\mathbb{N}$ of order-type ξ. Since a well-ordering is well-founded, the previous corollary yields an $x \in \mathcal{O}$ with $|x|_{\mathcal{O}} > \xi$ (namely $x = 2^{f(e)}$ for $R = R_e$). Hence ξ receives a notation and is thus constructive. □

Kleene's $\mathcal{O}$ is Π^1_1-complete

We now use the previous result to show that $\mathcal{O}$ is *many-one complete* for all Π^1_1 subsets of $\mathbb{N}$. First, we establish that $\mathcal{O}$ is in fact a Π^1_1 set.

Proposition 20.5: *$\mathcal{O}$ and $<_{\mathcal{O}}$ are Π^1_1 sets.*

Proof. First note that $\mathcal{O} = \mathrm{dom}(<_{\mathcal{O}})$ and

$$x \in \mathrm{dom}(<_{\mathcal{O}}) \quad \Leftrightarrow \quad \exists y[x <_{\mathcal{O}} y \ \vee \ y <_{\mathcal{O}} x].$$

Since Π^1_1 sets are closed under projection along $\mathbb{N}$, $\exists^{\mathbb{N}}$, $<_{\mathcal{O}}$ being Π^1_1 implies that $\mathcal{O}$ is Π^1_1.

Let h be recursive such that if $x \in \mathcal{O}$,

$$W_{h(x)} =$$

$\square$

Theorem 20.6: *For every Π^1_1 set $X \subseteq \mathbb{N}$ there exists a recursive function f such that*

$$x \in X \quad \Longleftrightarrow \quad f(x) \in \mathcal{O}.$$

Proof. By the Normal Form given at the beginning of this Lecture,

$$x \in X \quad \Longleftrightarrow \quad S(x) \text{ is well-founded.}$$

The tree $S(x)$ is recursive uniformly in x, so there exists a recursive function t such that $S(x) = R_{t(x)}$, where R_e is the eth recursively enumerable binary relation on $\mathbb{N}$. If we let f be a reduction from $\mathrm{WF}_{\mathbb{N}}$ to $\mathcal{O}$. Then

$$x \in X \quad \Longleftrightarrow \quad f(t(x)) \in \mathcal{O}.$$

$\square$

It is clear from the proof that $\mathrm{WF}_{\mathbb{N}}$ is also a Π^1_1 complete set.

Corollary 20.7: *$\mathcal{O}$ is not Σ^1_1.*

Proof. Similar to showing that WF is not Σ^1_1 – exhibit a Π^1_1 subset of $\mathbb{N}$ that is not Σ^1_1. This can be done using the universality of the Kleene T-predicate. $\square$

Lecture 21: Co-analytic Ranks

In the previous lecture we learned about how Π^1_1 set can be analyzed in terms of countable ordinals. In this lecture we will deepen this analysis. We will develop the theory of Π^1_1**-ranks**, which is a powerful tool in descriptive set theory. We can view the recursive function f that we constructed in the proof of Theorem 20.2 as the central fact:

$$\text{If } R_e \text{ is well-founded, then } \rho(R_e) \leq |f(e)|_{\mathcal{O}} \qquad (*)$$

Boundedness Principles

We start by picking up the observation made in Lemma 20.1. It states that r.e. subsets of $\mathcal{O}$ are **uniformly bounded**: Given an index e of an r.e. subset of $\mathcal{O}$, we can compute uniformly in e a ordinal bounding all ordinals denoted by W_e. We can strengthen this to Σ^1_1 sets.

Theorem 21.1 (Spector): *If $X \subseteq \mathcal{O}$ is Σ^1_1, then there exists $b \in \mathcal{O}$ such that*

$$\forall x \in X \quad |x|_{\mathcal{O}} < |b|_{\mathcal{O}}.$$

Proof. Let t be a reduction from $\mathcal{O}$ to $\mathrm{WF}_{\mathbb{N}}$, that is t is recursive such that

$$x \in \mathcal{O} \quad \Leftrightarrow \quad R_{t(x)} \text{ is well-founded.}$$

The idea is that if X is unbounded in $\mathcal{O}$, then we can characterize $\mathcal{O}$ by a Σ^1_1 formula, contradicting Corollary 20.7. If the desired b does not exist, then, for each $x \in \mathcal{O}$, we can find a $y \in X$ such that there exists an embedding of $R_{t(x)}$ into $\mathcal{O}$ below y. Using the proof of Theorem 20.2, we can formulate this as a property $P(x)$,

$$P(x) \quad \Leftrightarrow \quad \exists y\, [y \in X \;\wedge\; \exists \gamma \forall z_0, z_1\, (R_{t(x)}(z_0, z_1) \;\Rightarrow\; \langle \gamma(z_0), \gamma(z_1) \rangle \in W_{g(z)})],$$

where g is a recursive function so that $W_{g(z)} = \{\langle x, y \rangle : x <_{\mathcal{O}} y <_{\mathcal{O}} z\}$ (see Proposition 19.7). If X is Σ^1_1, then P is Σ^1_1.

If $x \in \mathcal{O}$, then $R_{t(x)}$ is well-founded, hence by $(*)$, $\rho(R_{t(x)}) \leq |f(t(x))|_{\mathcal{O}}$, and thus if X is unbounded in $\mathcal{O}$, $P(x)$ holds. If $P(x)$ holds on the other hand, then $R_{t(x)}$ must be well-founded (otherwise such a mapping would not exist), and thus $x \in \mathcal{O}$. Hence P would be a Σ^1_1 characterization of $\mathcal{O}$. $\square$

Corollary 21.2: *If $X \subseteq \mathbb{N}$ is Δ^1_1, and h is recursive such that $x \in X$ if and only if $h(x) \in \mathcal{O}$, then there exists a $b \in \mathcal{O}$ such that*

$$\forall x \in X \quad |h(x)| < |b|_{\mathcal{O}}.$$

A similar statement holds with $\mathrm{WF}_{\mathbb{N}}$ in place of $\mathcal{O}$.

Boundedness for sets of reals

The key to Spector's theorem is the fact that $\mathrm{WF}_{\mathbb{N}}$ and $\mathcal{O}$ are m-complete for the class of Π^1_1 sets of natural numbers.

We have seen (Theorem 17.6) that the set $\mathrm{WOrd}, \mathrm{WF} \subseteq \mathbb{N}^{\mathbb{N}}$ are $\boldsymbol{\Pi}^1_1$-complete with respect to *Wadge-reducibility*. This lets us obtain a similar result for Σ^1_1 sets of reals.

Theorem 21.3 (Σ^1_1-boundedness for reals)**:** *Let $A \subseteq \mathrm{WOrd}$ be Σ^1_1. Then there exists a $\xi < \omega_1^{\mathrm{CK}}$ such that*

$$\forall \alpha \in A \quad \|\alpha\| < \xi,$$

where $\|\alpha\|$ denotes the order type of the well-ordering coded by α.

An analogous statement holds for WF, with respect to the rank function ρ of a well-founded relation.

Proof. If such a ξ did not exist, then

$$\alpha \in \mathrm{WOrd} \quad \Longleftrightarrow \quad \exists \beta\, [\beta \in A \wedge \mathrm{WOrd}_\beta].$$

The right-hand side is Σ^1_1, and hence WOrd would be Σ^1_1, contradiction. □

Rank analysis of co-analytic sets

The previous results constitute a powerful technique when analyzing the complexity of sets. In particular, they give us a method to show that a Π^1_1 set is *not* Borel, besides proving that they are Π^1_1-complete.

If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^1_1, then there exists a recursive tree T such that

$$\alpha \in A \quad \Longleftrightarrow \quad T(\alpha) \text{ is well-founded.}$$

Every well-founded $T(\alpha)$ has a rank $\rho(T(\alpha))$. Σ^1_1-boundedness tells us that if A is moreover Δ^1_1, then the *spectrum* of these ranks is **bounded by a computable ordinal**. This means that we can show that A is not Δ^1_1 by showing that its ordinal spectrum $\{\rho(T(\alpha)) \colon \alpha \in A\}$ is unbounded in ω_1^{CK}.

These observations generalize (using relativization) to $\mathbf{\Pi}^1_1$ sets: Ranks of Borel sets are bounded by an ordinal $\xi < \omega_1$.

The downside of this method is that the tree T associated with a Π^1_1 set is a rather generic object, stemming from the canonical representation of Π^1_1 sets, and it may be rather difficult to prove anything about the ordinals $\rho(T(\alpha))$.

In many cases one can replace the canonical rank function with a "custom" one that better reflects the structure of a set.

Given a set S, a **rank** on S is a map $\varphi : S \to \mathrm{Ord}$. A rank is called **regular** if $\varphi(S)$ is an ordinal, i.e. $\varphi(S)$ is an initial segment of Ord.

Each rank gives rise to a **prewellordering** $\leq_\varphi$:

$$x \leq_\varphi y \quad \Leftrightarrow \quad \varphi(x) \leq \varphi(y).$$

A prewellordering is a binary relation on S that is reflexive, transitive, and *connected* (any two elements are comparable), and every non-empty subset of S has a $\leq_\varphi$-minimal element.

Under AC every set can be well-ordered, which means that every set admits a regular rank function that is one-one. However, we would like a rank function to reflect the complexity and structure of the set. In particular, we would like to preserve the boundedness properties of Σ^1_1 sets. For those to hold it was crucial that the initial segments WOrd_ξ, $\xi < \omega_1$ (and similarly for $\mathcal{O}$) are Borel.

We formulate a similar property that ensures the same for general rank functions.

Definition 21.4: Let X be a Polish space, and suppose $A \subseteq X$. A rank $\varphi : A \to \mathrm{Ord}$ is a **$\mathbf{\Pi}^1_1$-rank** if there exists a $\mathbf{\Sigma}^1_1$ relation $\leq^\Sigma_\varphi$ and a $\mathbf{\Pi}^1_1$ relation $\leq^\Pi_\varphi$ such that for $y \in A$,

$$\begin{aligned}\{x \in A \colon \varphi(x) \leq \varphi(y)\} &= \{x \in X : x \leq^\Sigma_\varphi y\} \\ &= \{x \in X : x \leq^\Pi_\varphi y\}.\end{aligned}$$

In other words, the initial segments $\leq_\varphi$ below a given $y \in A$ are uniformly $\mathbf{\Delta}^1_1$.

Theorem 21.5: *Every $\mathbf{\Pi}^1_1$ set $A \subseteq \mathbb{N}^{\mathbb{N}}$ admits a $\mathbf{\Pi}^1_1$-rank.*

Proof. We first show that WOrd admits a $\mathbf{\Pi}^1_1$-rank. The function φ is obviously $\varphi(\alpha) = \|\alpha\|$. We have to express $\|\alpha\| \leq \|\beta\|$ in a $\Sigma 1$ and a $\mathbf{\Pi}^1_1$ way.

For the $\mathbf{\Sigma}^1_1$ relation $\leq^\Sigma_\varphi$, let

$$
\begin{aligned}
\alpha \leq^\Sigma_\varphi \beta \quad &\Longleftrightarrow \quad E_\alpha \text{ is a linear ordering and} \\
&\qquad\ \ \exists\gamma\,[\gamma \text{ is a one-one, relation preserving mapping } \gamma : E_\alpha \to E_\beta] \\
&\Longleftrightarrow \quad E_\alpha \text{ is a linear ordering and } \exists\gamma\forall m,n\,[m\,E_\alpha\,n \Rightarrow \gamma(m)\,E_\beta\,\gamma(n)].
\end{aligned}
$$

Recall that "E_α is a linear ordering" is Π^0_1, hence $\leq^\Sigma_\varphi$ is Σ^1_1.

For the $\mathbf{\Sigma}^1_1$ relation $\leq^\Pi_\varphi$, let

$$
\begin{aligned}
\alpha \leq^\Pi_\varphi \beta \quad &\Longleftrightarrow \quad E_\alpha \text{ is a well-ordering and} \\
&\qquad\ \ \text{there is no relation preserving mapping of } E_\beta \text{ onto an initial segment of } E_\alpha \\
&\Longleftrightarrow \quad \alpha \in \mathrm{WOrd} \text{ and } \forall\gamma\neg\exists k\forall m,n\,[m\,E_\beta\,n \Rightarrow \gamma(m)\,E_\alpha\,\gamma(n)\,E_\alpha\,k].
\end{aligned}
$$

Since WOrd is Π^1_1, $\leq^\Pi_\varphi$ is Π^1_1, too.

Now we have for $\beta \in \mathrm{WOrd}$,

$$\alpha \leq^\Sigma_\varphi \beta \quad \Longleftrightarrow \quad \alpha \leq^\Pi_\varphi \beta \quad \Longleftrightarrow \quad \|\alpha\| \leq \|\beta\|,$$

as desired. □

Theorem 21.6 (Boundedness for arbitrary rank functions)**:** *Suppose $A \subseteq X$ is $\mathbf{\Pi}^1_1$ but not Borel and $\varphi : A \to \mathrm{Ord}$ is a $\mathbf{\Pi}^1_1$-rank on A. If $B \subseteq A$ is $\mathbf{\Sigma}^1_1$, then there is an $x_0 \in A$ such that*

$$\varphi(x) \leq \varphi(x_0) \quad \text{for all } x \in B.$$

Proof. If not, then

$$x \in A \quad \Longleftrightarrow \quad \exists y\,[y \in B \wedge x \leq^\Sigma_\varphi y],$$

and thus A would be $\mathbf{\Sigma}^1_1$, and thus Borel, a contradiction. □

Corollary 21.7: *Suppose $A \subseteq X$ is $\mathbf{\Pi}^1_1$ and $\varphi : A \to \mathrm{Ord}$ is a regular $\mathbf{\Pi}^1_1$-rank. Then*

(a) $\varphi(A) \leq \omega_1$;

(b) A is Borel if $\varphi(A) < \omega_1$;

(c) if $B \subseteq A$ is $\mathbf{\Sigma}^1_1$, then $\sup\{\varphi(x) : x \in B\} < \omega_1$.

The Cantor-Bendixson Rank

We illustrate the concept of $\mathbf{\Pi}^1_1$-ranks with a rank function that is different from the canonical rank function.

Suppose T is a tree on $\{0,1\}$. Define the **Cantor-Bendixson derivative** of T as

$$T' = \{\sigma \in T : \ \sigma \text{ has at least two incompatible extensions}\}.$$

We can iterate this derivative along the ordinals:

$$T^{(\xi+1)} = (T^{(\xi)})' \quad \text{and}$$
$$T^{(\lambda)} = \bigcup_{\xi<\lambda} T^{(\xi)} \quad \text{for } \lambda \text{ limit.}$$

We clearly have $T^{(\zeta)} \subseteq T^{(\xi)}$ for $\zeta < \xi$. There must exist an ordinal ξ_0 such that $(T^{(\xi_0)})' = T^{(\xi_0)}$. Since T is countable, $\xi_0 < \omega_1$. We call the least such ξ_0 the **Cantor-Bendixson rank** of T, $\| T \|_{\mathrm{CB}}$.

The following is not hard to see.

Proposition 21.8: *For any tree T,*

(a) if $[T^{\|T\|_{\mathrm{CB}}}] \neq \emptyset$, then $[T^{\|T\|_{\mathrm{CB}}}]$ is a perfect subset of $\mathbb{N}^{\mathbb{N}}$;

(b) $T^{\|T\|_{\mathrm{CB}}} = \emptyset$ if and only if $[T]$ is countable.

We hence have a new proof of the Cantor-Bendixson Theorem 2.5 for $2^{\mathbb{N}}$.

One can show that $\|\,.\,\|_{\mathrm{CB}}$ is indeed a $\mathbf{\Pi}^1_1$-rank on the set of all countable compact subsets of $2^{\mathbb{N}}$. This follows from the theory of **Borel derivatives**, which generalizes the Cantor-Bendixson derivative to other settings (see **(author?)** [Kec95]).

Since for any given ordinal $\xi < \omega_1$, we can find a tree $T \subseteq 2^{<\mathbb{N}}$ with $\| T \|_{\mathrm{CB}} = \xi$, it follows that the set

$$K_\omega(2^{\mathbb{N}}) = \{K \subseteq 2^{\mathbb{N}} : K \text{ countable}\}$$

is not Borel.

Using a different derivative, **(author?)** [KW86] showed that the set

$$\mathrm{Diff} = \{f \in \mathcal{C}[0,1] : f \text{ differentiable on } [0,1]\}$$

is not Borel.

Lecture 22: Hyperarithmetical Sets

Is there an effective counterpart to Souslin's Theorem that Borel $= \mathbf{\Delta}^1_1$? Definability in second order arithmetic gives us the lightface classes Σ^0_n and Π^0_n for finite n, but what would a "lightface" Σ^0_ξ set be?

Instead of definability, we can also describe the lightface classes using computational properties. In Lecture 9 we saw that the Σ^0_n and Π^0_n sets of reals correspond to the Borel sets of finite level with computable codes. We could try to extend the notion of a code into the transfinite, by introducing a "limit" of codes to deal with limit ordinals. A limit code should give us a method how to effectively recover codes for the sets whose union (limit) we are taking. This is, by no coincidence, reminiscent of the concept of an ordinal notation, where the limit notation was essentially an index for the notations of the ordinals whose limit we are taking.

The transfinite jump operation

We first illustrate the method for subsets of $\mathbb{N}$. Here we have the advantage that iterating definability corresponds directly to iterating the jump operator. So if we can define a transfinite extension of the Turing jump, this should give us a blueprint of how to define Σ^0_ξ, Π^0_ξ sets for infinite ordinals.

The H-sets

The template for a transfinite extension of the jump is given by

$$\begin{aligned} 0^{(n+1)} &= (0^{(n)})' \\ 0^{(\omega)} &= \{\langle n, m\rangle : m \in 0^{(n)}\}, \end{aligned}$$

that is, at limit stages we take effective unions of predecessors in the jump hierarchy. The predecessors are increasing in complexity and "lead up" to $O^{(\omega)}$. The general definition could read therefore something like

$$0^{(\xi)} = \{\langle n, m\rangle : n \text{ codes an ordinal } \zeta < \xi \text{ and } m \in 0^{(\zeta)}\}.$$

This clearly suggests to use ordinal notations.

Definition 22.1: For all $x \in \mathcal{O}$, we define the H**-set** H_x recursively as follows:

$$\begin{aligned} H_1 &= 0, \\ H_{2^x} &= (H_x)', \\ H_{3 \cdot 5^x} &= \{\langle y, m \rangle : y <_{\mathcal{O}} 3 \cdot 5^x \ \wedge \ m \in H_y\}. \end{aligned}$$

The rules obviously assign a strictly ascending sequence of Turing degrees with every **path** of $\mathcal{O}$, i.e. every subset of $\mathcal{O}$ that is linearly ordered by $<_{\mathcal{O}}$ and is closed downwards under $<_{\mathcal{O}}$. We will see later that notations of equal $|.|_{\mathcal{O}}$-rank define H-sets of equal Turing degree.

We start by observing that higher instances of H-sets can compute their predecessors (in a uniform way).

Proposition 22.2: *If $x, y \in \mathcal{O}$, and $x \leq_{\mathcal{O}} y$, then $H_x \leq_1 H_y$ uniformly in x, y (that is, an index for the reduction from H_x to H_y can be found uniformly in x, y).*

Proof. Let f be recursive such that for any $X \subseteq \mathbb{N}$, $X \leq_1 X'$ via f. Consider two cases:

$|y|_{\mathcal{O}} = |x|_{\mathcal{O}} + n$: We can simply take $h = f^n$, the n-fold iterate of f.

$|y|_{\mathcal{O}} \geq |x|_{\mathcal{O}} + \omega$: Let $z \in \mathcal{O}, n \in \mathbb{N}$ be such that $|z|_{\mathcal{O}}$ is limit and $|y|_{\mathcal{O}} = |z|_{\mathcal{O}} + n$. Then

$$m \in H_x \quad \Leftrightarrow \quad \langle x, m \rangle \in H_z \quad \Leftrightarrow \quad f^n(\langle x, m \rangle) \in H_y.$$

Note that f^n is one-one, that both cases can be distinguished effectively in x, y, and that n can be found effectively in x, y. $\square$

Next we show that the jump of an H-set can compute the ordinal notations below its rank. Given $x \in \mathcal{O}$, let

$$\mathcal{O}_x = \{y \in \mathcal{O} : |y|_{\mathcal{O}} < |x|_{\mathcal{O}}\}.$$

Proposition 22.3: *For each $x \in \mathcal{O}$, $\mathcal{O}_x \leq H_{2^x}$, uniformly in x.*

Proof. The proof is by effective transfinite recursion. One constructs a computable function f such that if $x \in \mathcal{O}$,

$$\mathcal{O}_x = \Phi_{f(x)}^{O_{2^x}}.$$

We sketch how to do the induction step, and leave the fully formalized argument (in the manner of Lecture 19) to the reader (see **(author?)** [Sac90]). We consider the following cases, which can be distinguished effectively:

$x = 1$: Then $\mathcal{O}_x = \emptyset$. In this case just let $f(x)$ be an index c such that $\Phi_c^X = \emptyset$ for any oracle X.

$x = 2^s$, $s = 2^t$: Then $\mathcal{O}_x = \mathcal{O}_s \cup \{2^y : y \in \mathcal{O}_s\}$. By assumption, $\mathcal{O}_s \leq_{\mathrm{T}} H_{2^s} = H_x$. It follows that $\mathcal{O}_x$ is recursive in H_x, too. An index for the reduction can be found uniformly from an index of the reduction $\mathcal{O}_s \leq_{\mathrm{T}} H_x$.

$x = 2^s$, $s = 3 \cdot 5^e$: In this case

$$\mathcal{O}_x = \mathcal{O}_s \cup \{3 \cdot 5^e : \varphi_e \text{ total and } \forall n\ \varphi_e(n) <_{\mathcal{O}} \varphi_e(n+1)\}.$$

Let q be a recursive function such that $W_{q(x)} = \{\langle y, z\rangle : y <_{\mathcal{O}} z <_{\mathcal{O}} x\}$. Let $X = \{3{\cdot}5^e : \varphi_e \text{ total and } \forall n\ \langle \varphi_e(n), \varphi_e(n{+}1)\rangle \in W_{q(x)}\}$. Then $\mathcal{O}_x = \mathcal{O}_s \cup X$. Again by hypothesis, $\mathcal{O}_s \leq_{\mathrm{T}} H_x$. Furthermore, $X \leq_{\mathrm{T}} 0'' = H_4$, and thus by Proposition 22.2, $X \leq_{\mathrm{T}} H_x$. The two reductions can be combined uniformly into a single reduction $\mathcal{O}_x \leq_{\mathrm{T}} H_x \leq_{\mathrm{T}} H_{2^x}$.

$x = 3 \cdot 5^s$: Then $\mathcal{O}_x = \{y : \exists n\ y \in \mathcal{O}_{\varphi_s(n)}\}$. By induction hypothesis, $\mathcal{O}_{\varphi_s(n)} \leq_{\mathrm{T}} H_{2^{\varphi_s(n)}}$, and by Proposition 22.2, $H_{2^{\varphi_s(n)}} \leq_{\mathrm{T}} H_x$ uniformly in n, x. Hence $\mathcal{O}_x$ is r.e. in H_x, and therefore recursive in H_{2^x}. Again, all reductions are uniform. (This case is where we need the jump of H_x to compute $\mathcal{O}_x$.)

□

If we want to compute the set of notations of the same rank as x, we need one more jump.

Corollary 22.4: *For any* $x \in \mathcal{O}$,

$$\mathcal{O}_{=x} = \{y \in \mathcal{O} : |y|_{\mathcal{O}} = |x|_{\mathcal{O}}\} \leq_{\mathrm{T}} H_{2^{2^x}},$$

uniformly in x.

Proof. We have

$$\mathcal{O}_{=x} = \mathcal{O}_{2^x} \setminus \mathcal{O}_x.$$

Apply the previous proposition. □

We are now in a position to show that the Turing degree of an H-set is invariant under passing to a notation of equal rank.

Theorem 22.5 (Spector)**:** *For any* $x, y \in \mathcal{O}$,

$$|x|_{\mathcal{O}} = |y|_{\mathcal{O}} \quad \Rightarrow \quad H_x \equiv_{\mathrm{T}} H_y,$$

uniformly in x, y.

Proof. The proof proceeds by effective transfinite recursion on the set

$$\mathcal{O}_= = \{\langle x, y\rangle : x, y \in \mathcal{O} \text{ and } |x|_\mathcal{O} = |y|_\mathcal{O}\},$$

along the well-founded relation $\langle s, t\rangle \prec \langle x, y\rangle$ which holds if and only if $|s|_\mathcal{O} < |x|_\mathcal{O}$. We sketch the recursion step for the construction of a reduction $H_x \leq_T H_y$ (the reduction $H_y \leq_T H_x$ is obtained in a completely analogous fashion), an consider the following cases:

$\langle x, y\rangle = \langle 1, 1\rangle$: Choose an index c such that $\Phi_c^\emptyset = \emptyset$.

$\langle x, y\rangle \in \mathcal{O}_=$, $\langle x, y\rangle = \langle 2^s, 2^t\rangle$: By induction hypothesis, $H_s \leq_T H_t$, and thus, by the monotonicity of the jump operator,

$$H_x = H_{2^s} = (H_s)' \leq_T (H_t)' = H_{2^t} = H_y,$$

and the reduction can be found uniformly.

$\langle x, y\rangle \in \mathcal{O}_=$, $\langle x, y\rangle = \langle 3\cdot 5^s, 3\cdot 5^t\rangle$: We want to decide whether $\langle z, m\rangle \in H_x$, given an oracle for H_y. For this, we have to decide whether $z <_\mathcal{O} x$ and $m \in H_z$.

By Proposition 19.7, the set of all $z <_\mathcal{O} x$ is r.e. and hence can be decided in $0' = H_2 \leq_T H_y$. For each such z, we have to decide whether $m \in H_z$. For this, enumerate all $v <_\mathcal{O} y$. Since $|y|_\mathcal{O} = |x|_\mathcal{O}$, we must eventually find a v such that $|v|_\mathcal{O} = |z|_\mathcal{O}$. The latter fact can be tested recursively in $H_{2^{2^v}}$, by Corollary 22.4. Since y is limit, $H_{2^{2^v}} \leq_T H_y$. Finally, by induction hypothesis H_v computes H_z, and by Proposition 22.2, $H_y \geq_T H_s$. All procedures described are uniform in x, y and an index for the uniform reduction up to $\langle x, y\rangle$.

□

Hyperarithmetic $= \Delta^1_1$

A set $X \subseteq \mathbb{N}$ is called **hyperarithmetic** if it is recursive in some H-set. We will see that the hyperarithmetic sets of natural numbers are precisely the Δ^1_1 definable sets, thereby giving an effective analog to Souslin's Theorem.

We first show that if X is hyperarithmetic, then X is Δ^1_1. We will actually show something stronger: Uniformly in $x \in \mathcal{O}$ we can compute a Δ^1_1-*index* for H_x.

The normal form for Π^1_1 sets discussed in Lecture 20 yields that for every Π^1_1 set X there exists an $e \in \mathbb{N}$ such that

$$x \in X \quad \Leftrightarrow \quad \forall \alpha \exists y\, [T(e, x, y, \alpha\restriction_y) \wedge U(y) = 0].$$

In this case e is called a Π^1_1**-index** for X. A Δ^1_1**-index** for a Δ^1_1 set Y is a number $d = \langle e_0, e_1 \rangle$ such that e_0 is a Π^1_1-index for Y and e_1 is a Π^1_1-index for $\mathbb{N} \setminus X$ (such an e_1 is also called a Σ^1_1**-index** for X).

Theorem 22.6 (Kleene)**:** *There exists a recursive function f such that if $x \in \mathcal{O}$, then $f(x)$ is a Δ^1_1-index for H_x.*

Proof. The proof proceeds as usual by effective transfinite recursion along $<_{\mathcal{O}}$. We skip the details and sketch how to do the recursion step.

$x = 1$**:** Let $f(x)$ be a Δ^1_1 index for the empty set.

$x = 2^s$**:** By induction hypothesis, we can assume we have constructed a function φ_c such that $\varphi_c(s)$ is a Δ^1_1-index of H_s. We have to show that effectively in x, c we can find a Δ^1_1-index for $(H_s)'$. Given any $X \subseteq \mathbb{N}$, $x \in X'$ if and only if $\Phi^X_x(x)\downarrow$, if and only if

$$\exists \sigma\; T(x, x, |\sigma|, \sigma) \wedge \sigma = X\restriction_{|\sigma|}.$$

We can use the Δ^1_1-index for X to express the last part of the formula in terms of a T-normal form of X and of $\mathbb{N} \setminus X$. We have to do this twice – for the Σ^1_1-index, and for the Π^1_1-index. Bringing the whole expression into T-normal form gives a Σ^1_1 index and a Π^1_1-index for X', respectively. (For details see **(author?)** [Sac90].)

$x = 3 \cdot 5^s$**:** We have $H_x = \{\langle y, m \rangle : y <_{\mathcal{O}} 3 \cdot 5^x \;\wedge\; m \in H_y\}$. By induction hypothesis, we have constructed a function φ_c such that

$$H_x = \{\langle y, m \rangle : y <_{\mathcal{O}} 3 \cdot 5^x \wedge \forall \alpha \exists m [T((\varphi_c(x))_0, x, m, \alpha\restriction_m) \wedge U(m) = 0]\}.$$

$y <_{\mathcal{O}} 3 \cdot 5^s$ is uniformly r.e. in s. Normalizing yields a Π^1_1-index for H_x. A Σ^1_1-index is obtained similarly.

□

Corollary 22.7: *If $X \subseteq \mathbb{N}$ is hyperarithmetic, then it is Δ^1_1.*

Proof. The Δ^1_1 sets are closed downward under Turing reducibility. (Exercise!) □

Finally, we show that being Δ^1_1 implies being hyperarithmetic. This is an intriguing consequence of the boundedness principle.

Theorem 22.8 (Kleene)**:** *If $X \subseteq \mathbb{N}$ is Δ^1_1, then X is hyperarithmetic.*

Proof. If X is Δ^1_1, then it is many-one reducible to $\mathcal{O}$. Let g be recursive such that

$$x \in X \quad \Leftrightarrow \quad g(x) \in \mathcal{O}.$$

Now define $Z \subseteq \mathbb{N}$ by

$$z \in Z \quad \Leftrightarrow \quad \exists y \, [y \in X \, \wedge \, z = g(y)].$$

$Z = g(X)$ is a Σ^1_1 subset of $\mathcal{O}$, and by Spector's Boundedness Theorem 21.1 there exists $b \in \mathcal{O}$ such that

$$\forall z \in Z \quad |z|_{\mathcal{O}} < |b|_{\mathcal{O}}.$$

This means

$$x \in X \quad \Leftrightarrow \quad g(x) \in \mathcal{O}_b.$$

By Proposition 22.3, $\mathcal{O}_b$ is recursive in H_{2^b}, and hence X is hyperarithmetic. $\square$

Let HYP be the set of hyperarithmetic sets of natural numbers.

Corollary 22.9: *The set* HYP *is a* Π^1_1 *subset of* $2^{\mathbb{N}}$.

Proof. We have

$$X \in \mathrm{HYP} \quad \Leftrightarrow \quad \exists x \, [x \in \mathcal{O} \, \wedge \, X \leq_{\mathrm{T}} H_x].$$

Since $\mathcal{O}$ is Π^1_1, H_x has a (uniformly) Δ^1_1 definition, and Turing reducibility can be expressed via an arithmetical formula, the result follows. $\square$

References

[Gro99] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.

[Hal50] Paul R. Halmos. *Measure Theory*. Van Nostrand, 1950.

[Jec03] Thomas Jech. *Set Theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[Kan03] Akihiro Kanamori. *The Higher Infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.

[Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*. Springer, 1995.

[KW86] Alexander S. Kechris and W. Hugh Woodin. Ranks of differentiable functions. *Mathematika*, 33(2):252–278 (1987), 1986.

[Rog87] Hartley Rogers, Jr. *Theory of recursive functions and effective computability*. MIT Press, Cambridge, MA, 1987.

[Sac90] Gerald E. Sacks. *Higher Recursion Theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990.

[Sol70] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. of Math. (2)*, 92:1–56, 1970.

[Ury27] P. Urysohn. Sur un espace métrique universel. I, II. *Bull. Sci. Math.*, 51:1–38, 1927.