

# INDEPENDENCE, RELATIVE RANDOMNESS, AND PA DEGREES

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ABSTRACT. We study pairs of reals that are mutually Martin-Löf random with respect to a common, not necessarily computable probability measure. We show that a generalized version of van Lambalgen's Theorem holds for non-computable probability measures, too. We study, for a given real  $A$ , the *independence spectrum* of  $A$ , the set of all  $B$  so that there exists a probability measure  $\mu$  so that  $\mu\{A, B\} = 0$  and  $(A, B)$  is  $\mu \times \mu$ -random. We prove that if  $A$  is r.e., then no  $\Delta_2^0$  set is in the independence spectrum of  $A$ . We obtain applications of this fact to PA degrees. In particular, we show that if  $A$  is r.e. and  $P$  is of PA degree so that  $P \not\leq_T A$ , then  $A \oplus P \geq_T \emptyset'$ .

## 1. INDEPENDENCE AND RELATIVE RANDOMNESS

The property of independence is central to probability theory. Given a probability space with measure  $\mu$ , we call two measurable sets  $\mathcal{A}$  and  $\mathcal{B}$  independent if

$$\mu\mathcal{A} = \frac{\mu(\mathcal{A} \cap \mathcal{B})}{\mu\mathcal{B}}.$$

The idea behind this definition is that if event  $\mathcal{B}$  occurs, it does not make event  $\mathcal{A}$  any more or less likely. This paper considers a similar notion, that of relative randomness. The theory of algorithmic randomness provides a means of defining which elements of Cantor space ( $2^\omega$ ) are random. We call  $A \in 2^\omega$  Martin-Löf random if  $A$  is not an element of any effective null set. We denote the class of all Martin-Löf random reals by  $\text{MLR}$ .<sup>1</sup>

We say that  $A$  is Martin-Löf random relative to  $B$ , or  $A \in \text{MLR}(B)$  if  $A$  is not an element of any null set effective in  $B$ . Relative randomness is analogous to independence because if  $A \in \text{MLR}(B)$ , then not only is  $A$  a random real but *even given* the information in  $B$ , we cannot capture  $A$  in an effective null set. If we start with the assumption that  $A$  and  $B$  are both Martin-Löf random, then the following theorem of van Lambalgen establishes that relative randomness is symmetrical.

**Theorem 1.1** (Van Lambalgen [1987]). *If  $A, B \in \text{MLR}$  then  $A \in \text{MLR}(B)$  if and only if  $B \in \text{MLR}(A)$  if and only if  $A \oplus B \in \text{MLR}$ .*

We can extend the notion of relative randomness to any probability measure. We take  $\mathcal{P}(2^\omega)$  to be the set of all Borel probability measures on

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<sup>1</sup>For a comprehensive presentation of the theory of Martin-Löf randomness, see the monographs by Downey and Hirschfeldt [2010] and Nies [2009].

Cantor space. Endowed with the weak-\* topology,  $\mathcal{P}(2^\omega)$  becomes a compact metrizable space. The measures that are a finite, rational-valued, linear combination of Dirac measures form a countable dense subset, and one can choose a metric on  $\mathcal{P}(2^\omega)$  that is compatible with the weak-\* topology so that the distance between those basic measures is a computable function, and with respect to which  $\mathcal{P}(2^\omega)$  is complete. In other words,  $\mathcal{P}(2^\omega)$  can be given the structure of an *effective Polish space*. We can represent measures via Cauchy sequences of basic measures. This allows for coding measures as reals, and one can show that there exists a continuous mapping  $\rho : 2^\omega \rightarrow \mathcal{P}(2^\omega)$  so that for any  $X \in 2^\omega$ ,

$$\rho^{-1}(\{\rho(X)\}) \text{ is a } \Pi_1^0(X) \text{ class.}$$

For details of this argument, see Day and Miller [ta]. If  $\mu \in \mathcal{P}(2^\omega)$ , any real  $R$  with  $\rho(R) = \mu$  is called a *representation* of  $\mu$ .

We want to define randomness relative to a parameter with respect to a probability measure  $\mu$ . Martin-Löf's framework easily generalizes to tests that have access to an oracle. However, our test should have access to *two* sources: the parameter of relative randomness and the measure (in form of a representation).

**Definition 1.2.** Let  $R_\mu$  be a representation of a measure  $\mu$ , and let  $A \in 2^\omega$ .

- (a) A  $(R_\mu, A)$ -test is given by a sequence  $(\mathcal{V}_n : n \in \mathbb{N})$  of uniformly  $\Sigma_1^0(R_\mu \oplus A)$ -classes  $\mathcal{V}_n \subseteq 2^\omega$  such that for all  $n$ ,  $\mu(\mathcal{V}_n) \leq 2^{-n}$ .
- (b) A real  $X \in 2^\omega$  passes an  $(R_\mu, A)$ -test  $(\mathcal{V}_n)$  if  $X \notin \bigcap_n \mathcal{V}_n$ .
- (c) A real  $X \in 2^\omega$  is  $(R_\mu, A)$ -random if it passes all  $(R_\mu, A)$ -tests.

If, in the previous definition,  $A = \emptyset$ , we simply speak of an  $R_\mu$ -test and of  $X$  being  $R_\mu$ -random.

The previous definition defines randomness with respect to a specific representation. If  $X$  is random for one representation, it is not necessarily random for other representations. On the other hand, we can ask whether a real exhibits randomness with respect to *some* representation, so the following definition makes sense.

**Definition 1.3.** A real  $X \in 2^\omega$  is  $\mu$ -random relative to  $A \in 2^\omega$ , or simply  $\mu$ - $A$ -random if there exists a representation  $R_\mu$  of  $\mu$  so that  $X$  is  $(R_\mu, A)$ -random. We denote by  $\text{MLR}_\mu(A)$  the set of all  $\mu$ - $A$ -random reals.

For Lebesgue measure  $\lambda$ , we sometimes suppress the measure. Hence, in accordance with established notation,  $\text{MLR}(A)$  denotes the set of all Martin-Löf random reals.

A most useful property of the theory of Martin-Löf randomness is the existence of *universal tests*. Universal tests subsume all other tests. Furthermore, they can be defined uniformly with respect to any parameter. The construction can be extended to tests with respect to a measure  $\mu$ . More precisely, there exists a uniformly c.e. sequence  $(U_n : n \in \mathbb{N})$  of sets  $U_n \subseteq 2^{<\omega}$  such that, if we set for  $R, A \in 2^\omega$

$$\mathcal{U}_n^{R,A} = \{[\sigma] : \langle \sigma, \tau_0, \tau_1 \rangle \in U_n, \tau_0 \prec R, \tau_1 \prec A\},$$

then  $(\mathcal{U}_n^{R,A})$  is an  $(R, A)$ -test and  $X \in 2^\omega$  is  $(R, A)$ -random if and only if  $X \notin \bigcap_n \mathcal{U}_n^{R,A}$ . We call  $(U_n)$  a *universal oracle test*.

Since for any  $R \in 2^\omega$ ,  $\rho^{-1}(\rho(R))$  is  $\Pi_1^0(R)$ , we can eliminate the representation of a measure in a test for randomness by defining, for any  $A \in 2^\omega$ ,

$$\tilde{\mathcal{U}}_n^{R,A} = \bigcap_{S \in \rho^{-1}(\{\rho(R)\})} \mathcal{U}_n^{S,A}.$$

The resulting class  $\tilde{\mathcal{U}}_n^A$  is still  $\Sigma_1^0(R)$ , since  $\rho^{-1}(\{\rho(R)\})$  is  $\Pi_1^0(R)$  and hence compact.

**Proposition 1.4.** *For any  $R, A \in 2^\omega$  with  $\rho(R) = \mu$ , a real  $X$  is  $\mu$ - $A$ -random if and only if*

$$X \notin \bigcap_n \tilde{\mathcal{U}}_n^{R,A}.$$

*Proof.* If  $X$  is  $\mu$ - $A$ -random, then it passes every  $(R_\mu, A)$ -test for some representation  $R_\mu$  of  $\mu$ , in particular the instance  $(\mathcal{U}_n^{R_\mu, A})$  of the universal oracle test. Since  $R_\mu \in \rho^{-1}(\{\rho(R)\})$ , it follows that  $X$  passes  $\tilde{\mathcal{U}}_n^{R,A}$ .

On the other hand, if for every representation  $R_\mu$  of  $\mu$ ,  $X$  fails the test  $(\mathcal{U}_n^{R_\mu, A})$ , then  $X \in \bigcap_n \tilde{\mathcal{U}}_n^{R,A}$ .  $\square$

The previous proposition shows that the test  $\tilde{\mathcal{U}}_n^{R,A}$  is related to the concept of a *uniform test*, originally introduced by Levin [1976], and further developed by Gács [2005] and Hoyrup and Rojas [2009]. Hence we call it a *uniform oracle test*. Note that if  $R, S$  are both representations of a measure  $\mu$ , then the uniform oracle tests  $(\tilde{\mathcal{U}}_n^{R,A})_n$  and  $(\tilde{\mathcal{U}}_n^{S,A})_n$  are identical.

**Definition 1.5.** Take  $A, B \in 2^\omega$  and  $\mu \in \mathcal{P}(2^\omega)$ . We say that  $A$  and  $B$  are *relatively random with respect to  $\mu$*  if  $A \in \text{MLR}_\mu(B)$  and  $B \in \text{MLR}_\mu(A)$ .

Note that the representations of  $\mu$  witnessing randomness for  $A$  and  $B$ , respectively, do not have to be identical. If  $A$  and  $B$  are relatively random with respect to some measure  $\mu$ , then  $\mu$  might offer some information about the relationship between  $A$  and  $B$ . For example, we know that if  $A$  and  $B$  are relatively random with respect to Lebesgue measure, then any real they both compute must be K-trivial. If  $A$  and  $B$  are both atoms of  $\mu$  then clearly  $A$  and  $B$  are relatively random with respect to  $\mu$ . Given this, perhaps the most obvious question to ask about relative randomness is the following.

**Question 1.6.** For which  $A, B \in 2^\omega$  does there exist a measure  $\mu$  such that  $A$  and  $B$  are relatively random with respect to  $\mu$  and neither  $A$  nor  $B$  is an atom of  $\mu$ ?

This question is closely related to a theorem of Reimann and Slaman [2008]. They proved that an element  $X$  of Cantor space is non-recursive if and only if there exists a measure  $\mu$  such that  $X$  is  $\mu$ -random and  $X$  is not an atom of  $\mu$ .

Van Lambalgen's theorem shows that  $A$  and  $B$  are relatively random if and only if  $A \oplus B \in \text{MLR}$ . If we take  $\lambda$  to be the uniform measure, then  $A \oplus B \in \text{MLR}$  if and only if the pair  $(A, B) \in 2^\omega \times 2^\omega$  is Martin-Löf random with respect to the product measure  $\lambda \times \lambda$  i.e.  $(A, B) \in \text{MLR}_{\lambda \times \lambda}$ . We begin our investigation into relative randomness by showing that van

Lambalgen's theorem holds for any Borel probability measure on Cantor space.

**Theorem 1.7.** *Let  $\mu \in \mathcal{P}(2^\omega)$  and let  $A, B \in 2^\omega$  then  $(A, B) \in \text{MLR}_{\mu \times \mu}$  if and only if  $A \in \text{MLR}_\mu$  and  $B \in \text{MLR}_\mu(A)$ .*

*Proof.* Let  $R$  be any representation of  $\mu$ . First let us consider if  $B \notin \text{MLR}_\mu(A)$ . In this case we have that  $B \in \bigcap_n \mathcal{U}_n^{R,A}$ . We define an  $(R, \emptyset)$ -test for  $2^\omega \times 2^\omega$  by  $\mathcal{V}_n^R = \{[\tau] \times [\sigma] : \exists \eta \prec R (\langle \sigma, \eta, \tau \rangle \in U_n)\}$ . This ensures that  $(A, B) \in \bigcap_n \mathcal{V}_n^R$ . By applying Fubini's theorem we can establish that:

$$\begin{aligned} (\mu \times \mu)(\mathcal{V}_n^R) &= \int_{2^\omega \times 2^\omega} \chi_{\mathcal{V}_n^R}(X, Y) d\mu \times d\mu \\ &= \int_{2^\omega} \left( \int_{2^\omega} \chi_{\mathcal{U}_n^{R,X}}(Y) d\mu(Y) \right) d\mu(X) \\ &\leq \int_{2^\omega} 2^{-n} d\mu(X) = 2^{-n} \end{aligned}$$

Hence  $(A, B)$  is not  $(R, \emptyset)$ -random. As this is true for any representation  $R$  of  $\mu$  we have that  $(A, B) \notin \text{MLR}_{\mu \times \mu}$ . The same argument shows *a fortiori* that if  $A \notin \text{MLR}_\mu$  then  $(A, B) \notin \text{MLR}_{\mu \times \mu}$ .

To establish the other direction assume that  $(A, B) \notin \text{MLR}_{\mu \times \mu}$ . Again let  $R$  be any representation of  $\mu$ .

Hence  $(A, B) \in \bigcap_n \mathcal{V}_n^R$ , where  $(\mathcal{V}_n^R)$  is a universal  $R$ -test for  $2^\omega \times 2^\omega$ . Let

$$\mathcal{W}_n^{R,X} = \{Y : (X, Y) \in \mathcal{V}_n^R\}$$

We have that  $\mathcal{W}_n^{R,X}$  is a  $\Sigma_1^0(R \oplus X)$  class and this is uniform in  $n$ . However, given any  $X$ , we do not know whether or not  $\mu(\mathcal{W}_n^{R,X}) \leq f(n)$  for some decreasing recursive function  $f$  such that  $\lim_n f(n) = 0$ . Hence we cannot necessarily turn this into a Martin-Löf test relative to  $X$ . In fact it is not even necessarily true that  $\liminf_n \mu(\mathcal{W}_n^{R,X}) = 0$ . We will show that the failure to turn this into a Martin-Löf test for some  $X \in 2^\omega$  implies that  $X \notin \text{MLR}_\mu$ . This is a slight strengthening of the result that van Lambalgen obtained in his thesis. Van Lambalgen showed that if  $\liminf_n \mu(\mathcal{W}_n^{R,X}) \neq 0$  then  $X \notin \text{MLR}_\mu$ .

However, we can generalize the proof of van Lambalgen's theorem given in Nies [2009]. We define another  $R$ -test by letting  $\mathcal{T}_n^R = \{X \in 2^\omega : \mu(\mathcal{W}_{2n}^{R,X}) > 2^{-n}\}$ . To see that  $\mathcal{T}_n^R \leq 2^{-n}$ , note that

$$\begin{aligned} (\mu \times \mu)\mathcal{V}_{2n}^R &\geq \int_{\mathcal{T}_n^R \times 2^\omega} \chi_{\mathcal{V}_{2n}^R}(X, Y) d\mu \times d\mu \\ &= \int_{\mathcal{T}_n^R} \int_{2^\omega} \chi_{\mathcal{W}_{2n}^{R,X}}(Y) d\mu(Y) d\mu(X) \\ &\geq \int_{\mathcal{T}_n^R} 2^{-n} d\mu(X) = 2^{-n} \mu(\mathcal{T}_n^R) \end{aligned}$$

Now as  $2^{-2n} \geq (\mu \times \mu)\mathcal{V}_{2n}^R$ , we have that  $\mu(\mathcal{T}_n^R) \leq 2^{-n}$ . Hence  $\bigcap_n \mathcal{T}_n^R$  is an  $R$ -test. Assume that  $A \notin \text{MLR}_\mu(R)$ . Then  $A$  avoids all but finitely many of the sets  $\mathcal{T}_n^R$ . Hence for all but finitely many  $n$  we have that  $\mu \mathcal{W}_{2n}^{R,A} \leq 2^{-n}$

and so by modifying finitely many  $\mathcal{W}_{2^n}^{R,A}$  we can obtain an  $(R, A)$ -test that covers  $B$ . Therefore  $B$  is not  $R$ -random relative to  $A$ .

For all representations  $R$  of  $\mu$ , we have shown that either  $A$  is not  $R$ -random or  $B$  is not  $R$ -random relative to  $A$ . However, to prove the theorem, it is essential that we get the *same* outcome for all representations i.e. if  $(A, B) \notin \text{MLR}_\mu$  then either for all representations  $R$  of  $\mu$ ,  $A$  is not  $R$ -random or for all representations  $R$  of  $\mu$ ,  $B$  is not  $R$ -random relative to  $A$ .

We can resolve this problem by taking our test  $(\mathcal{V}_n^R)$  on the product space to be a uniform test. In this case we always obtain the same “projection tests”  $(\mathcal{W}_n^{R,X})$  (independent of  $R$ ) and hence the same outcome for any representation of  $\mu$ .  $\square$

**Corollary 1.8.** *If  $A, B \in 2^\omega$  and  $\mu \in \mathcal{P}(2^\omega)$ , then  $A$  and  $B$  are relatively random with respect to  $\mu$  if and only if  $(A, B) \in \text{MLR}_{\mu \times \mu}$ .*

**Corollary 1.9.** *If  $A \geq_{\text{T}} B$  and  $(A, B) \in \text{MLR}_{\mu \times \mu}$  then  $B$  must be an atom of  $\mu$ .*

*Proof.* This holds because  $B \in \text{MLR}_\mu(A)$  if and only if  $B$  is an atom of  $\mu$ .  $\square$

We note that we cannot extend one direction of van Lambalgen’s theorem to product measures of the form  $\mu \times \nu$ . In particular it is not true that if  $A \in \text{MLR}_\mu$  and  $B \in \text{MLR}_\nu(A)$  then  $(A, B) \in \text{MLR}_{\mu \times \nu}$ . For example, it is possible to code a real  $B$  into a measure  $\mu$  in such a way that  $B$  is computable from every representation of  $\mu$  and choose a measure  $\nu$  so that there exist  $A \in \text{MLR}_\mu$ ,  $B \in \text{MLR}_\nu(A)$ , with  $\mu\{A\} = \nu\{B\} = 0$ . But for any such  $\mu, \nu$  we cannot have  $(A, B) \in \text{MLR}_{\mu \times \nu}$ , since any  $\mu \times \nu$ -test has access to  $\mu$  and hence can compute  $B$ .

Given any  $X \in 2^\omega$ , we will use  $\mathcal{R}(X)$  to denote the set of reals  $Y$  such that  $X$  and  $Y$  are relatively random with respect to some measure  $\mu$  and neither  $X$  nor  $Y$  are atoms of  $\mu$  i.e.

$$\mathcal{R}(A) = \{B \in 2^\omega : (\exists \mu \in \mathcal{P}(2^\omega))[(A, B) \in \text{MLR}_{\mu \times \mu} \text{ and } \mu\{A, B\} = 0]\}.$$

We call  $\mathcal{R}(A)$  the *independence spectrum* of  $A$ .

The following proposition lists some basic properties of the independence spectrum.

**Proposition 1.10.** *For all  $A, B \in 2^\omega$  the following hold:*

- (1)  $A \in \mathcal{R}(B)$  if and only if  $B \in \mathcal{R}(A)$ .
- (2)  $B \in \mathcal{R}(A)$  implies that  $A \upharpoonright_{\text{T}} B$ .
- (3) If  $A$  is non-recursive and  $\nu$  is a computable, non-atomic measure (i.e. a measure with a computable representation and  $\nu\{X\} = 0$  for all  $X \in 2^\omega$ ), then  $\mathcal{R}(A)$  has  $\nu$ -measure 1.
- (4) If  $A \in \text{MLR}$  then  $\text{MLR}(A) \subsetneq \mathcal{R}(A)$ .

*Proof.* (1) is by definition and (2) is by Corollary 1.9.

(3) Suppose  $A$  is non-recursive and  $\nu$  is a computable measure with  $\nu\{A\} = 0$ . There is a measure  $\mu$  such that  $A$  is not an atom of  $\mu$  and

$A \in \text{MLR}_\mu$ , say via a representation  $R_\mu$ . Let  $\kappa = (\mu + \nu)/2$ . There exists a representation  $R_\kappa \leq_T R_\mu$ , as  $\nu$  is computable. We claim that  $A$  is  $R_\kappa$ -random. For if not, then  $A$  fails some  $R_\kappa$ -test  $(\mathcal{W}_n^{R_\kappa})$ . We have

$$\mu \mathcal{W}_n^{R_\kappa} = 2\kappa \mathcal{W}_n^{R_\kappa} - \nu \mathcal{W}_n^{R_\kappa} \leq 2\kappa \mathcal{W}_n^{R_\kappa} \leq 2^{n-1}.$$

Since  $R_\kappa \leq_T R_\mu$ ,  $(\mathcal{W}_{n+1}^{R_\kappa})$  would define an  $R_\mu$ -test that covers  $A$ , contradicting the assumption that  $A$  is  $R_\mu$ -random. Furthermore, by assumption on  $\mu$  and  $\nu$ ,  $\kappa\{A\} = 0$ . Hence

$$(\text{MLR}_\kappa(A) \setminus \{B : \kappa\{B\} \neq 0\}) \subseteq \mathcal{R}(A),$$

by van Lambalgen's Theorem.

Now  $\nu(\text{MLR}_\kappa(A)) = 1$  because the complement of  $\text{MLR}_\nu(A)$  is a  $\kappa$  null set and hence a  $\nu$  null set ( $\nu$  is absolutely continuous with respect to  $\kappa$  by definition). Moreover, the set of atoms of  $\kappa$  is countable and so has  $\nu$ -measure 0 by the assumption that  $\nu$  is non-atomic. This gives us that

$$\nu(\text{MLR}_\kappa(A) \setminus \{B : \kappa\{B\} \neq 0\}) = 1$$

and thus  $\nu\mathcal{R}(A) = 1$ .

(4) Suppose  $A$  is Martin-Löf random. By the definition of  $\mathcal{R}(A)$  and Theorem 1.7 we have that  $\text{MLR}(A) \subseteq \mathcal{R}(A)$ .

On the other hand,  $A$  is not recursive and hence by (3),  $\mathcal{R}(A)$  has measure 1 for any computable, non-atomic measure. Let  $\nu$  be a computable, non-atomic measure orthogonal to Lebesgue measure (e.g. the  $(1/3, 2/3)$ -Bernoulli measure). Since  $\nu\mathcal{R}(A) = 1$ ,  $\mathcal{R}(A)$  has to contain a  $\nu$ -random element  $X$ . But  $X$  cannot be relatively Martin-Löf random. Therefore,  $\text{MLR}(A) \subsetneq \mathcal{R}(A)$ .  $\square$

The proposition shows that, outside the upper and lower cone of a real  $A$ , the complement of  $\mathcal{R}(A)$  is rather small measure wise. On the other hand, the above properties leave open the possibility that  $\mathcal{R}(A)$  is just the set of reals that are Turing incomparable with  $A$ . We will now establish that this is not necessarily the case.

**Proposition 1.11.** *Let  $R$  be a representation of a measure  $\mu$ . If  $A \in 2^\omega$  is such that*

- (1)  $A$  is r.e.<sup>2</sup>,
- (2)  $A$  is  $R$ -random, and
- (3)  $A$  is not an atom of  $\mu$ ,

then  $R \oplus A \geq_T R'$ .

*Proof.* Given such an  $R$  and  $A$ , let  $A_s$  be a recursive approximation to  $A$ . We define the function  $f \leq_T A \oplus R$  by:

$$f(x) = \min\{s : (\exists m \leq s)(A_s \upharpoonright m = A \upharpoonright m \wedge \mu_s[A \upharpoonright m] < 2^{-x})\}.$$

In this definition we take  $\mu_s[\sigma]$  to be an  $R$ -recursive approximation to  $\mu[\sigma]$  from above. Note that  $f$  is well defined because  $A$  is not an atom of  $\mu$ . We claim that if  $g$  is any partial function recursive in  $R$ , then for all but finitely many  $x \in \text{dom}(g)$ , we have that  $f(x) > g(x)$ . To establish this claim, let

<sup>2</sup>We mean here, of course, that  $A$  is recursively enumerable viewed as a subset of  $\mathbb{N}$ , by identifying a subset of  $\mathbb{N}$  with the real given by its characteristic sequence.

$g$  be an  $R$ -recursive partial function. We will build an  $R$ -test  $\{U_n\}_{n \in \omega}$  by defining  $U_n$  to be:

$$\{X \in 2^\omega : (\exists x > n)(\exists m)(g(x) \downarrow \wedge \mu[A_{g(x)} \upharpoonright m] < 2^{-x} \wedge X \succ (A_{g(x)} \upharpoonright m))\}.$$

Because any  $x \in \text{dom}(g)$  adds a single open set  $([A_{g(x)} \upharpoonright m])$  for some  $m$  of measure less than  $2^{-x}$  to those  $U_n$  with  $n < x$ , we have constructed a valid test. Now if  $g(x) \downarrow \geq f(x)$ , then by definition of  $f$ , there is some  $m \leq f(x)$  such that  $\mu[A \upharpoonright m] < 2^{-x}$  and  $A \upharpoonright m = A_{f(x)} \upharpoonright m = A_{g(x)} \upharpoonright m$ . Thus for all  $n < x$ ,  $A \in U_n$ . Because  $A \in \text{MLR}_\mu(R)$  we have that  $f(x) > g(x)$  for all but finitely many  $x$  in  $\text{dom}(g)$ .

Let  $g(x)$  be the  $R$ -recursive partial function with domain  $R'$  such that  $g(x)$  is the unique  $s$  such that  $x \in R'_{s+1} \setminus R'_s$ . For almost all  $x$ , we have that  $x \in R'$  if and only if  $x \in R'_{f(x)}$  and so  $R' \leq_T A \oplus R$ .  $\square$

**Theorem 1.12.** *Let  $R$  be a representation of a measure  $\mu$ . If*

- (1)  $A$  is r.e.,
- (2)  $A$  is  $\mu$ -random, and
- (3)  $A$  is not an atom of  $\mu$ ,

then  $R \oplus A \geq_T \emptyset'$ .

*Proof.* Note the following characteristics of the previous proof. First the totality of  $f$  does not depend on the fact that  $A$  is  $R$ -random, it only depends on the fact that  $A$  is not an atom of  $\mu$ . The construction is uniform so there is a single index  $e$  such that  $\Phi_e(A \oplus \hat{R})$  is total if  $\hat{R}$  is any representation of  $\mu$ . Additionally if  $A$  is  $\hat{R}$ -random then for all but finitely many  $x$ ,  $\Phi_e(A \oplus \hat{R}; x) \geq g(x)$  where  $g$  is any  $\hat{R}$ -recursive partial computable function.

Let  $R$  be any representation of  $\mu$ . The set  $\{A \oplus \hat{R} : \hat{R} \text{ is a representation of } \mu\}$  is a  $\Pi_1^0(A \oplus R)$  class and  $\Phi_e$  is total on this class. From  $A \oplus R$  we can compute a function  $f$  that dominates  $\Phi_e(A \oplus \hat{R})$  where  $A$  is  $\hat{R}$ -random. As  $f$  dominates any  $\hat{R}$ -recursive partial function we have that  $A \oplus R \geq_T \emptyset'$ .  $\square$

**Corollary 1.13.** *If  $A$  is r.e. and  $B \leq_T \emptyset'$  then  $B \notin \mathcal{R}(A)$ .*

The question remains, however, how big the independence of a real can be outside its upper and lower cones.

**Question 1.14.** Is the set of all  $X$  so that  $X \upharpoonright_T A$  and  $X \notin \mathcal{R}(A)$  countable?

## 2. RECURSIVELY ENUMERABLE SETS AND PA DEGREES

We will now give two (somehow unexpected) applications of Theorem 1.12 to the interaction between recursively enumerable sets and sets of PA degree. Recall that a set  $A \subseteq \mathbb{N}$  is of *PA degree* if it is Turing equivalent to a set coding a complete extension of Peano arithmetic (PA). PA degrees have many interesting computability theoretic properties. For instance, a set is of PA degree if and only if it computes a path through every non-empty  $\Pi_1^0$  class. However, a complete degree-theoretic characterization of the PA degrees is still not known. If  $A \geq_T \emptyset'$ , then  $A$  is of PA degree. On the other hand, Gödel's First Incompleteness Theorem implies that no r.e. set can be

a complete extension of PA. Jockusch and Soare [1972] showed moreover that if a set is of incomplete r.e. degree, it cannot be of PA degree.

It seems therefore worthwhile to gain a complete understanding how r.e. sets and PA degrees are related. The crucial fact that links Theorem 1.12 to PA degrees is a result by Day and Miller [ta]. They showed that every set of PA degree computes a representation of a *neutral measure*. Such a measure has the property that *every* real is random with respect to it, i.e.  $2^\omega = \text{MLR}_\mu$ . The existence of neutral measures was first established by Levin [1976].

Our first result shows that below  $\emptyset'$ , r.e. sets and PA degrees behave quite complementary<sup>3</sup>.

**Corollary 2.1** (to Theorem 1.12). *If  $A$  is an r.e. set and  $P$  a set of PA degree such that  $P \not\geq_T A$  then  $P \oplus A \geq_T \emptyset'$ .*

*Proof.* By the result of Day and Miller [ta] mentioned above,  $P$  computes a representation  $R_\mu$  of a neutral measure  $\mu$  and  $A \in \text{MLR}_\mu$ . Day and Miller [ta] also showed that a real  $X$  is an atom of a neutral measure if every representation of the measure computes  $X$ . Now because  $P \not\geq_T A$ , we have that  $A$  is not an atom of  $\mu$ . Thus all hypotheses of Theorem 1.12 are satisfied and we have  $P \oplus A \geq_T \emptyset'$ .  $\square$

Corollary 2.1 strengthens a result due to Kučera and Slaman (unpublished). Recall that a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is *diagonally non-recursive* if  $f(n) \neq \varphi_n(n)$  for all  $n$ , where  $\varphi_n$  denotes, as usual, the  $n$ th partial recursive function. Kučera and Slaman constructed a low<sub>2</sub> r.e. set so that  $A \oplus f \equiv_T \emptyset'$  for any diagonally non-recursive function  $f \leq_T \emptyset'$ . It is well-known that every PA degree computes a  $\{0, 1\}$ -valued diagonally non-recursive function. Hence the set constructed by Kučera and Slaman joins any PA degree below  $\emptyset'$  to  $\emptyset'$ . Corollary 2.1 yields that this is in fact true for *any* r.e. set.

One can now ask which kind of incomplete r.e. sets *can* be bounded by PA degrees below  $\emptyset'$ . This question was first raised by Kučera [2004]

*For which incomplete r.e. sets  $A$  does there exist set  $P$  of PA degree such that  $A <_T P <_T \emptyset'$ ?*

We can use Corollary 2.1 to completely answer this question. We say a set  $B$  is of *PA degree relative* to a set  $A$ , written  $B \gg A$  (see [Simpson, 1977]), if  $B$  computes a path through every  $\Pi_1^0(A)$  class. One well-known fact we will make use of is the following. If  $P$  is of PA degree, then there exists a set  $Q$  of PA degree such that  $P \gg Q$ . One way to prove this fact is to observe that the  $\Pi_1^0$  class

$$\{(A, B) \in 2^\omega \times 2^\omega : A \in \text{DNR}_2 \wedge B \in \text{DNR}_2(A)\}$$

is non-empty, where  $\text{DNR}_2$  and  $\text{DNR}_2(A)$  are the classes of  $\{0, 1\}$ -valued diagonally non-recursive functions and  $\{0, 1\}$ -valued diagonally non-recursive functions relative to  $A$ , respectively.

**Theorem 2.2.** *If  $A$  is an r.e. set then the following are equivalent:*

<sup>3</sup>After the authors announced the result presented in Corollary 2.1, proofs not involving measure theoretic arguments have been found independently by A. Kučera and J. Miller.



- (1)  $A$  is low.
- (2) There exists  $P$ ,  $P \gg A$  and  $P$  is low.
- (3) There exists  $P$  of PA degree such that  $\emptyset' \geq_T P \geq_T A$ .

*Proof.* (1)  $\Rightarrow$  (2): There is a (non-empty)  $\Pi_1^0(A)$  class of sets  $B \gg A$ . Relativize the low basis theorem to find  $P \gg A$  and  $P' \equiv_T A'$ . As  $A$  is low so is  $P$ .

(2)  $\Rightarrow$  (3): This is clear.

(3)  $\Rightarrow$  (1): Take any  $Q$  of PA degree such that  $P \gg Q$ . Now  $Q \geq_T A$  because otherwise  $Q \oplus A \geq \emptyset'$  but this is impossible because  $P \geq_T Q \oplus A$  and  $P \not\geq_T \emptyset'$ . Hence  $P \gg A$ . But now we have that  $\emptyset'$  is r.e. in  $A$  and also  $\emptyset'$  computes a DNR function relative to  $A$ . Hence by relativizing Arslanov's completeness criterion we have that  $A' \equiv_T \emptyset'$ .  $\square$

Observe that in the proof of (3)  $\Rightarrow$  (1), showing  $P \gg A$  only used the facts that  $P \not\geq_T \emptyset'$  and  $P \geq_T A$ . Hence we get a final corollary.

**Corollary 2.3.** *If  $P$  is a set of PA degree and  $A$  is an r.e. set such that  $P \geq_T A$  and  $P \not\geq_T \emptyset'$ , then  $P \gg A$ .*

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