SCHNORR DIMENSION

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ABSTRACT. Following Lutz's approach to effective (constructive) dimension, we define a notion of dimension for individual sequences based on Schnorr's concept(s) of randomness. In contrast to computable randomness and Schnorr randomness, the dimension concepts defined via computable martingales and Schnorr tests coincide, i.e. the Schnorr Hausdorff dimension of a sequence always equals its computable Hausdorff dimension. Furthermore, we give a machine characterization of Schnorr dimension, based on prefix-free machines whose domain has computable measure. Finally, we show that there exist computably enumerable sets which are Schnorr (computably) irregular: while every c.e. set has Schnorr Hausdorff dimension 0 there are c.e. sets of computable packing dimension 1, a property impossible in the case of effective (constructive) dimension, due to Barzdiņš' Theorem. In fact, we prove that every hyperimmune Turing degree contains a set of computable packing dimension 1.

1. INTRODUCTION

Schnorr [22] issued a fundamental criticism concerning the notion of effective null sets introduced by Martin-Löf. He argued that, although we know how fast a Martin-Löf test converges to zero, it is not effectively given, in the sense that the measure of the test sets U_n is not computable, but only enumerable from below, so in general we cannot decide whether a given cylinder belongs to the nth level of some test.

Schnorr presented two alternatives, both clearly closer to what one would call a *computable approach to randomness*. One is based on the idea of randomness as an unpredictable event in the sense that it should not be possible to win in a betting game (martingale) against a truly random sequence of outcomes. The other sticks to Martin-Löf's approach, but requires the tests defining a null set to be *a uniformly computable sequence* of open sets having uniformly *computable measure*, not merely a sequence of uniformly computably enumerable sets such that the nth set has measure less than 2^{-n}.

Schnorr was able to show that both approaches yield reasonable notions of randomness, i.e. random sequences according to his concepts exhibit most of the robust properties one would expect from a random object. However, his suggestions have some serious drawbacks. They are harder to deal with technically, which is mainly due to the absence of universal tests. Besides, a machine characterization of randomness like the elegant coincidence of Martin Löf-random sequences with those incompressible by a universal prefix-free machine is technically more involved and was only recently given by [6].

Recently, [12, 13, 14] introduced an effective notion of Hausdorff dimension. As (classical) Hausdorff dimension can be seen as a refinement of Lebesgue measure on 2^ω, in the sense that it further distinguishes between classes of measure 0, effective Hausdorff dimension of an individual sequence can be interpreted as a *degree of randomness* of the sequence. This viewpoint is supported by a series of results due to [19, 20], [25, 27], [3], and [15], which establish that the effective Hausdorff dimension of a sequence equals its lower asymptotic Kolmogorov complexity (plain or prefix-free).

Lutz's framework of martingales (gales) is very flexible as regards the level of effectivization one wishes to obtain, see [12]. Therefore, it is easy to define a version of algorithmic dimension based on computable martingales, *computable dimension*, in analogy to computable randomness. This has been done in [12], and has been briefly treated in [30].

In this paper we will study the Schnorr-style approach to algorithmic dimension in more detail. We will define a notion of dimension based on Schnorr's test concept. We will see that the technical difficulties mentioned above apply to dimension as well. Furthermore, Schnorr dimension in many respects behaves like effective (constructive) dimension, introduced by [13, 14] (see also [16, 17]). However, we will also see that for dimension, Schnorr's two approaches coincide, in contrast to Schnorr randomness and computable randomness: The Schnorr Hausdorff dimension of a sequence always equals its computable Hausdorff dimension. Furthermore, it turns out that, with respect to Schnorr/computable dimension, computably enumerable sets can exhibit a complex behavior, to some extent. Namely, we will show that there are c.e. sets of high computable packing dimension, which is impossible in the effective case, due to a result by [2]. In fact, every hyperimmune Turing degree contains a set of computable packing dimension 1 and this set can be chosen to be c.e. in the special case of a c.e. Turing degree. On the other hand, we prove that the computable Hausdorff dimension of the characteristic sequence of a c.e. set is 0. Thus, the class of computably enumerable sets contains *irregular* sequences – sequences for which Hausdorff and packing dimension do not coincide.

The paper is structured as follows: In Section 2 we give a short introduction to the classical theory of Hausdorff measures and dimension, as well as packing dimension. In Section 3 we will define algorithmic variants of these concepts based on Schnorr's test approach to randomness.

In Section 4 we prove that the dimension concepts based on Schnorr tests on the one hand and computable martingales on the other hand coincide, in contrast to Schnorr randomness and computable randomness. We also present two basic examples of sequences of nonintegral dimension (Section 5). In Section 6 we derive a machine characterization of Schnorr/computable Hausdorff and packing dimension. Finally, in Section 7, we study the Schnorr/computable dimension of computably enumerable sets. The main result here will be that on those sets computable Hausdorff dimension and computable packing dimension can differ as largely as possible.

We will use fairly standard notation. 2^ω will denote the set of infinite binary sequences. Sequences will be denoted by upper case letters like A, B, C, or X, Y, Z. We will refer to the nth bit ($n \geq 0$) in a sequence B by either B_n or $B(n)$, i.e. $B = B_0 B_1 B_2 \ldots = B(0)\, B(1)\, B(2) \ldots$.

Strings, i.e. finite sequences of 0s and 1s, will be denoted by lower case letters from the end of the alphabet, u, v, w, x, y, z along with some lower case Greek letters like σ and τ. $2^{<\omega}$ will denote the set of all strings. ϵ denotes the empty string. The *initial segment of length n*, $A\restriction_n$, of a sequence A is the string of length n corresponding to the first n bits of A. More generally, if $Z \subseteq \mathbb{N}$, we let $A\restriction_Z$ denote the restriction of A to the elements of Z. Formally, if $Z = \{z_0, z_1, \ldots\}$ with $z_n < z_{n+1}$

$$A\restriction_Z (n) = A(z_n).$$

This way, $A\restriction_n = A\restriction_{\{0,\ldots,n-1\}}$.

Given two strings v, w, the string v is called a *prefix* of w, $v \sqsubseteq w$ for short, if there exists a string x such that $vx = w$, where vx is the concatenation of v and x. If w is strictly longer than v, we write $v \sqsubset w$, and we extend this notation in a natural way to pairs of a string and a sequence. A set of strings is called *prefix-free* if no element of the set has a prefix (other than itself) in the set.

Initial segments induce a standard topology on 2^ω. The basis of the topology is formed by the *basic open cylinders* (or just *cylinders*, for short). Given a string $w = w_0 \dots w_{n-1}$ of length n, the basic open cylinder corresponding to w is defined as

$$[w] = \{A \in 2^\omega : A\upharpoonright_n = w\}.$$

We extend this notation to sets of strings: Given $C \subseteq 2^{<\omega}$, we define

$$[C] = \bigcup_{w \in C} [w].$$

Throughout the paper we assume familiarity with the basic concepts of computability theory such as Turing machines, computably enumerable sets, computable and left-computable (or left-c.e. or just c.e.) reals, and some central concepts of algorithmic information theory, in particular Kolmogorov complexity and the Kraft-Chaitin Theorem. A standard reference for computability theory is [23], while a comprehensive treatise of algorithmic information theory is [11]. The forthcoming book by [7] will cover both areas.

2. Hausdorff Measures and Dimension

The basic idea behind Hausdorff dimension is to determine which "scaling factor" reflects best the geometry of a set. One devises a family of (outer) measures, the so-called Hausdorff measures, which are generalizations of Lebesgue measure in the sense that they introduce a parameter with which the open sets (or rather their diameters) used in a covering are scaled. These measures are linearly ordered by the scaling parameter, and the Hausdorff dimension of a set picks out the parameter which induces the most "suitable" measure.

Definition 1. Let $\mathcal{X} \subseteq 2^\omega$.

(1) Given $\delta > 0$, a set $C \subseteq 2^{<\omega}$ is a *δ-cover* of $\mathcal{X}$ if

$$(\forall w \in C)\,[2^{-|w|} \leq \delta] \quad \text{and} \quad \mathcal{X} \subseteq [C].$$

(2) For $s \geq 0$, define

$$\mathcal{H}^s_\delta(\mathcal{X}) = \inf\left\{\sum_{w \in C} 2^{-|w|s} : C \text{ is a } \delta\text{-cover of } \mathcal{X}\right\}.$$

The *s-dimensional Hausdorff measure* of $\mathcal{X}$ is defined as

$$\mathcal{H}^s(\mathcal{X}) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(\mathcal{X}).$$

Remark. Given a string w, the cylinder $[w]$ has diameter $2^{-|w|}$ according to a standard metric compatible with the cylinder topology. Hence, the s-Hausdorff measure is obtained by restricting the admissible covers to finer and finer diameters. This is a geometric condition, and Hausdorff measures form an essential part of the theory of *fractal geometry*. Note further that $\mathcal{H}^s(\mathcal{X})$ is well defined, since, as δ decreases, there are fewer δ-covers available, hence $\mathcal{H}^s_\delta$ is nondecreasing. However, the value may be infinite. It can be shown that $\mathcal{H}^s$ is an outer measure and that the Borel sets of 2^ω are $\mathcal{H}^s$-measurable. For $s = 1$, one obtains Lebesgue measure on 2^ω.

The outer measures $\mathcal{H}^s$ have an important property.

Proposition 1. *Let $\mathcal{X} \subseteq 2^\omega$. If, for some $s \geq 0$, $\mathcal{H}^s(\mathcal{X}) < \infty$, then $\mathcal{H}^t(\mathcal{X}) = 0$ for all $t > s$.*

Proof. Let $\mathcal{H}^s(\mathcal{X}) < \infty$, $t > s$. If $C \subseteq 2^{<\omega}$ is a δ-cover of $\mathcal{X}$, $\delta > 0$, we have

$$\sum_{w\in C} 2^{-|w|t} \leq \delta^{t-s} \sum_{w\in C} 2^{-|w|s},$$

so, taking infima, $\mathcal{H}^t_\delta(\mathcal{X}) \leq \delta^{t-s}\mathcal{H}^s_\delta(\mathcal{X})$. As $\delta \to 0$, the result follows. □

This means that there exists a point $s \geq 0$ where the s-dimensional Hausdorff measure drops from a positive (possibly infinite) value to zero. This point is the *Hausdorff dimension* of the class.

Definition 2. For a class $\mathcal{X} \subseteq 2^\omega$, define the *Hausdorff dimension* of $\mathcal{X}$ as

$$\dim_{\mathrm{H}} \mathcal{X} = \inf\{s \geq 0 : \ \mathcal{H}^s(\mathcal{X}) = 0\}.$$

It is not hard to show that the notion of Hausdorff dimension is well behaved: It is *monotone* (i.e. $\mathcal{X} \subseteq \mathcal{Y}$ implies $\dim_{\mathrm{H}}(\mathcal{X}) \leq \dim_{\mathrm{H}}(\mathcal{Y})$), and *stable*: If $\{\mathcal{X}_i\}_{i\in\mathbb{N}}$ is a countable family of classes, then

$$\dim_{\mathrm{H}}\Big(\bigcup_{i\in\mathbb{N}} \mathcal{X}_i\Big) = \sup_{i\in\mathbb{N}}\big\{\dim_{\mathrm{H}} \mathcal{X}_i\big\}.$$

Furthermore, it can be seen as a *refinement of measure* 0. If $\mathcal{X}$ has positive (outer) Lebesgue measure, then $\dim_{\mathrm{H}}(\mathcal{X}) = 1$ (as Lebesgue measure λ corresponds to $\mathcal{H}^1$). In particular, $\mathcal{H}^1(2^\omega) = \lambda(2^\omega) = 1$. On the other hand, no $\mathcal{X} \subseteq 2^\omega$ can have Hausdorff dimension greater than 1, as $\mathcal{H}^s(\mathcal{X}) = 0$ for all $s > 1$. Hence, classes of non-integral Hausdorff dimension are necessarily Lebesgue null classes.

We give two examples of classes of non-integral dimension.

Theorem 1. *(1) Let $Z \subseteq \mathbb{N}$ be such that* $\lim_n |Z \cap \{0, \dots, n-1\}|/n = \delta$. *Define*

$$\mathcal{D}_Z = \{A \in 2^\omega : \ A(n) = 0 \textit{ for all } n \notin Z\}.$$

Then $\dim_{\mathrm{H}} \mathcal{D}_Z = \delta$.

(2) For $s \in [0, 1]$, define $\mathcal{B}_s \subseteq 2^\omega$ as

$$\mathcal{B}_s = \left\{A \in 2^\omega : \ \lim_{n\to\infty} \frac{|\{k < n : \ A(k) = 1\}|}{n} = s\right\}.$$

Then $\dim_{\mathrm{H}} \mathcal{B}_s = -[s \log s + (1-s)\log(1-s)]$.

The first assertion is a special case of a general behavior of Hausdorff dimension under Hölder transformations, see [9]. The second result is due to [8]. For more on Hausdorff measures and dimension refer to [9].

2.1. **Packing dimension.** Packing dimension can be seen as a dual to Hausdorff dimension. While Hausdorff measures are defined in terms of coverings, that is, enclosing a set from outside, packing measures approximate from the inside, by packing the set "as densely as possible" with disjoint sets of small size.

For this purpose, we say that a prefix-free set $P \subseteq 2^{<\omega}$ is a *packing* in $\mathcal{X} \subseteq 2^\omega$, if for every $\sigma \in P$ there is an $X \in \mathcal{X}$ such that $\sigma \sqsubset X$. Geometrically speaking, a packing in $\mathcal{X}$ is a collection of mutually disjoint open balls with centers in $\mathcal{X}$. If the balls all have radius $\leq \delta$, we call it a δ-packing in $\mathcal{X}$.

Now one can try to find a packing "as dense as possible": Given $s \geq 0$, $\delta > 0$, let

$$\mathcal{P}^s_\delta(\mathcal{X}) = \sup \left\{ \sum_{w \in P} 2^{-|w|s} : P \text{ is a } \delta\text{-packing in } \mathcal{X}. \right\}. \tag{1}$$

As $\mathcal{P}^s_\delta(\mathcal{X})$ decreases with δ, the limit

$$\mathcal{P}^s_0(\mathcal{X}) = \lim_{\delta \to 0} \mathcal{P}^s_\delta(\mathcal{X})$$

exists. However, this definition leads to problems concerning stability: Taking, for instance, the rational numbers in the unit interval, we can find denser and denser packings yielding that for every $0 \leq s < 1$, $\mathcal{P}^s_0(\mathbb{Q} \cap [0,1]) = \infty$, hence this notion lacks countable additivity, in particular it is not a measure. This can be overcome by applying a Caratheodory-type process to $\mathcal{P}^s_0$. Define

$$\mathcal{P}^s(\mathcal{X}) = \inf \left\{ \sum \mathcal{P}^s_0(\mathcal{X}_i) : \mathcal{X} \subseteq \bigcup_{i \in \mathbb{N}} \mathcal{X}_i \right\}, \tag{2}$$

where the infimum is taken over arbitrary countable covers of $\mathcal{X}$. $\mathcal{P}^s$ is an (outer) measure on 2^ω, and it is Borel regular. $\mathcal{P}^s$ is called, in correspondence to Hausdorff measures, the *s-dimensional packing measure* on 2^ω. Packing measures were introduced in [31] and [29]. They can be seen as a dual concept to Hausdorff measures, and behave in many ways similarly to them. In particular, one may define *packing dimension* in the same way as Hausdorff dimension.

Definition 3. The *packing dimension* of a set $\mathcal{X} \subseteq 2^\omega$ is defined as

$$\dim_{\mathrm{P}} \mathcal{X} = \inf\{s : \mathcal{P}^s(\mathcal{X}) = 0\} = \sup\{s : \mathcal{P}^s(\mathcal{X}) = \infty\}. \tag{3}$$

It is not hard to see that the packing dimension of a set is always at least as large as its Hausdorff dimension. Once more we refer to [9] for details on packing measures and dimension.

2.2. **Martingales.** It is possible do characterize Hausdorff and packing dimension via *martingales*, too. This was a fundamental observation first by [12, 13] in the case Hausdorff dimension, and then by [1] for packing dimension.

Martingales have become a fundamental tool in probability theory. In Cantor space 2^ω, they can be understood as simple betting games, which is reflected in the following definition.

Definition 4. (a) A *betting strategy* b is a function $b : 2^{<\omega} \to [0,1] \times \{0,1\}$.

(b) Given a betting strategy b and a positive real number $\alpha > 0$ the *martingale* $d^\alpha_b : 2^{<\omega} \to [0,\infty)$ *induced by* b and α is inductively defined by $d^\alpha_b(\epsilon) = \alpha$, where ϵ denotes the empty string, and

$$d^\alpha_b(w i_w) = d^\alpha_b(w)(1 + q_w), \tag{4}$$

$$d^\alpha_b(w(1 - i_w)) = d^\alpha_b(w)(1 - q_w) \tag{5}$$

for $w \in 2^{<\omega}$ and $b(w) = (q_w, i_w)$. If $\alpha = 1$, the martingale d^α_b is *normed*.

(c) A *martingale* d is a function $\{0,1\}^* \to [0,\infty)$ that is induced by a betting strategy and some number $\alpha > 0$.

Martingales can be interpreted as capital functions of the accordant betting strategy, when applied to a binary sequence: $d(w)$ is the player's capital after bits $w(0), \dots, w(|w|-1)$ have been revealed to him.

It is easy to check that every martingale satisfies a *fairness condition*:

$$d(w) = \frac{d(w0) + d(w1)}{2} \quad \text{for all } w \in 2^{<\omega}.$$

This means that the betting game underlying a martingale is *fair* in the sense that the expected payoff is equal to the current capital.

Later we will use the fact that martingales are *additive*: If d_1 and d_2 are martingales, so is $d_1 + d_2$. Furthermore, if $(d_n)_{n\in\mathbb{N}}$ is a countable sequence of martingales, then $\sum_{n\in\mathbb{N}} 2^{-n} d_n(\epsilon)^{-1} d_n$ is a martingale, too.

The notions of Hausdorff and packing measure zero on 2^ω can be characterized through martingales. The smaller s gets, the harder it is to cover a given set in terms of s-dimensional Hausdorff/packing measure. This is reflected by the following winning condition for martingales.

Definition 5. Let $s \geq 0$, and d be a martingale.

(a) d is s-*successful* (s-*succeeds*) on a sequence $B \in 2^\omega$ if

$$d(B\restriction_n) \geq 2^{(1-s)n} \quad \text{for infinitely many } n. \tag{6}$$

(b) d is *strongly* s-*successful* (or s-*succeeds strongly*) on a sequence $B \in 2^\omega$ if

$$d(B\restriction_n) \geq 2^{(1-s)n} \quad \text{for all but finitely many } n. \tag{7}$$

The next theorem states that the relation between $\mathcal{H}^s$-null sets and s-successful martingales is indeed very close.

Theorem 2. *Let $\mathcal{X} \subseteq 2^\omega$. Then it holds that*

$$\dim_{\mathrm{H}} \mathcal{X} = \inf\{s : \text{ some martingale } d \text{ is } s\text{-successful on all } B \in \mathcal{X}\}. \tag{8}$$

$$\dim_{\mathrm{P}} \mathcal{X} = \inf\{s : \text{ some martingale } d \text{ is strongly } s\text{-successful on all } B \in \mathcal{X}\}. \tag{9}$$

In the form presented here, equation (8) was first proved by [12]. However, a close connection between Hausdorff dimension and winning conditions on martingales has been observed by [21] and [26]. Equation (8) is due to [1].

Note that, if a martingale s-succeeds on a sequence A, for any $t > s$ it will hold that

$$\limsup_{n\to\infty} \frac{d(A\restriction_n)}{2^{(1-t)n}} = \infty. \tag{10}$$

So, when it comes to dimension, we will, if convenient, use (10) and the original definition interchangeably. Furthermore, a martingale which satisfies (10) for $s = 1$ is simply called *successful on A*.

3. Schnorr Null Sets and Schnorr Dimension

We now define a notion of dimension based on Schnorr's test approach to randomness. The basic idea is to extend the concept of a Schnorr test to Hausdorff measures and show that an effective version of Proposition 1 holds. Then the definition of Schnorr Hausdorff dimension follows in a straightforward way. A definition of Schnorr packing dimension based on packing measures defined via coverings would be rather involved. However, in the next section we will see that Schnorr Hausdorff dimension can be characterized via martingales. In fact, Schnorr Hausdorff dimension and computable Hausdorff dimension coincide. This justifies to regard computable packing dimension, defined in terms of strongly successful computable martingales, as the dual to Schnorr Hausdorff dimension.

As we are mostly interested in algorithmic notions of dimension, it suffices to deal with rational valued dimensions s only. This way we do not have to worry about problems of effectivity concerning real numbers.

Definition 6. Let $s \geq 0$ be a rational number.

(a) A *Schnorr s-test* is a uniformly c.e. sequence $(S_n)_{n\in\mathbb{N}}$ of sets of strings which satisfies, for all n, the following conditions:
 (1) For all n,

$$\sum_{w\in S_n} 2^{-|w|s} \leq 2^{-n}. \tag{11}$$

 (2) The real number $\sum_{w\in S_n} 2^{-|w|s}$ is uniformly computable in n; that is, there exists a computable function f such that for each n, i, $\left|f(n,i) - \sum_{w\in S_n} 2^{-|w|s}\right| \leq 2^{-i}$.

(b) A class $\mathcal{A} \subseteq 2^\omega$ is *Schnorr s-null* if there exists a Schnorr s-test (S_n) such that

$$\mathcal{A} \subseteq \bigcap_{n\in\mathbb{N}} [S_n].$$

To be compatible with the conventional notation, we denote the Schnorr 1-null sets simply as *Schnorr null*. The *Schnorr random* sequences are those which are (as a singleton in 2^ω) not Schnorr null.

In [6] it is observed that, by adding elements, one can replace any Schnorr 1-test by an equivalent one (i.e., one defining the same Schnorr null sets) where each level of the test has measure exactly 2^{-n}. We can apply the same argument in the case of arbitrary rational s, and hence we may, if appropriate, assume that (11) holds with equality. (In this case condition (2) in Definition 6 is automatically satisfied.)

Note further that, for rational s, the sets S_n in a Schnorr s-test are actually uniformly computable, since to determine whether $w \in S_n$ it suffices to enumerate S_n until the accumulated sum given by $\sum 2^{-|w|s}$ exceeds $2^{-n} - 2^{|w|s}$ (assuming the measure of the n-th level of the test is in fact 2^{-n}). If w has not been enumerated so far, it cannot be in S_n. The converse, however, does not hold: If $W \subseteq 2^{<\omega}$ is computable, this does not necessarily imply that the measure of $[W]$ is computable.

One can describe Schnorr s-null sets also in terms of *Solovay tests*. Solovay tests were introduced in [24] and allowed for a characterization of Martin-Löf null sets via a single test set, instead of a uniformly computable sequence of test sets.

Definition 7. Let $s \geq 0$ be rational.

(a) A *Solovay s-test* is a c.e. set $D \subseteq 2^{<\omega}$ such that

$$\sum_{w\in D} 2^{-|w|s} \leq 1$$

(b) A Solovay s-test is *total* if

$$\sum_{w\in D} 2^{-|w|s}$$

is a computable real number.

(c) A Solovay s-test D *covers* a sequence $A \in 2^\omega$ if

$$(\exists^\infty w \in D)\,[w \sqsubset A].$$

In this case we also say that A *fails* the test D.

Theorem 3. *For any rational $s \geq 0$, a class $\mathcal{X} \subseteq 2^\omega$ is Schnorr s-null if and only if there is a total Solovay s-test which covers every sequence $A \in \mathcal{X}$.*

Proof. (⇒) Let $\mathcal{X}$ be Schnorr s-null via a test $(U_n)_{n\in\mathbb{N}}$. Let

$$C = \bigcup_{n\geq 1} U_n.$$

Obviously, C is a Solovay s-test which covers all of $\mathcal{X}$, so it remains to show that C is total. But in order to compute $c = \sum_{v\in C} 2^{-|v|s}$ with precision 2^{-n}, it suffices to compute, for $i = 1, \dots, n+1$, the measure of U_i up to precision $2^{-(i+n+1)}$.

(⇐) Let C be a total Solovay s-cover of $\mathcal{X}$. Given n, compute $c = \sum_{v\in C} 2^{-|v|s}$ up to precision 2^{-n-2}, say as a value $\widetilde{c}$. Now find a finite subset $\widetilde{C} \subseteq C$ such that

$$\widetilde{c} - 2^{-n-1} \leq \sum_{w\in\widetilde{C}} 2^{-|w|s} \leq \widetilde{c} - 2^{-n-2}.$$

Then $C \setminus \widetilde{C}$ covers every sequence $A \in \mathcal{X}$. Furthermore, it holds that $\sum_{w\in C\setminus\widetilde{C}} 2^{-|w|s} \leq 3/2^{n+2} \leq 1/2^n$. Hence, if we define $U_n = C \setminus \widetilde{C}$, the (U_n) will form a Schnorr s-test for $\mathcal{X}$. □

Note that the equivalence between Solovay and Schnorr s-tests does not extend to Martin-Löf s-tests in general. For a Martin-Löf s-test we only require the first condition (1) in Definition 6 but not the second one. Martin-Löf s-tests and the corresponding dimension notions have been explicitly studied by [28], [16], and [4]. Implicitly, via martingales, they were already present in Lutz's introduction of effective dimension [13]. Solovay showed that a set $\mathcal{X} \subseteq 2^\omega$ is covered by a Martin-Löf 1-test if and only if it is covered by a Solovay 1-test. However, [18] have recently shown that for any rational $0 < s < 1$ there exists a sequence A which is not Martin-Löf s-null but is covered by a Solovay s-test.

3.1. **Schnorr dimension.** Like in the classical case, for each class, the family of Schnorr s-measures possesses a critical value.

Proposition 2. *Let $\mathcal{X} \subseteq 2^\omega$. Then for any rational $s \geq 0$, if $\mathcal{X}$ is Schnorr s-null then it is also Schnorr t-null for any rational $t \geq s$.*

Proof. It suffices to show that if $s \leq t$, then every Schnorr s-test (U_n) is also a Schnorr t-test. So assume $\{U_n\}$ is a Schnorr s-test. Given any real $\alpha \geq 0$ and $l \in \mathbb{N}$, let

$$m_n(\alpha) := \sum_{w\in U_n} 2^{-|w|\alpha} \quad \text{and} \quad m_n^l(\alpha) := \sum_{\substack{w\in U_n\\|w|\leq l}} 2^{-|w|\alpha}.$$

It is easy to check that

$$m_n^l(t) \leq m_n(t) \leq m_n^l(t) + m_n(s)2^{(s-t)l}.$$

Now $m_n(s)$ is computable, as is $2^{(s-t)l}$, and $2^{(s-t)l}$ goes to zero as l gets larger. Therefore, we can effectively approximate $m_n(t)$ to any desired degree of precision. □

The definition of Schnorr Hausdorff dimension now follows in a straightforward way.

Definition 8. The *Schnorr Hausdorff dimension* of a class $\mathcal{X} \subseteq 2^\omega$ is defined as

$$\dim_{\mathrm{H}}^{\mathrm{S}} \mathcal{X} = \inf\{s \geq 0 : \mathcal{X} \text{ is Schnorr } s\text{-null}\}.$$

For a sequence $A \in 2^\omega$, we write $\dim_{\mathrm{H}}^{\mathrm{S}} A$ for $\dim_{\mathrm{H}}^{\mathrm{S}}\{A\}$ and refer to $\dim_{\mathrm{H}}^{\mathrm{S}} A$ as the Schnorr Hausdorff dimension of A.

Note that the Schnorr Hausdorff dimension of any sequence is at most 1, since for any $\varepsilon > 0$ the "trivial" $1+\varepsilon$-test $W_n = \{w : |w| = l_n\}$, $l_n \geq \lceil n/\varepsilon \rceil$, will cover all of 2^ω.

3.2. **Schnorr packing dimension.** Due to the more involved definition of packing dimension, it is not immediately clear how to define a Schnorr-type version of packing dimension. However, we will see in the next section that Schnorr Hausdorff dimension allows an elegant characterization in terms of martingales, building on Theorem 2.

4. Schnorr Dimension and Martingales

In view of his unpredictability paradigm for algorithmic randomness, [22] suggested a notion of randomness based on *computable* martingales. According to this notion, nowadays referred to as *computable randomness*, a sequence is computably random if no computable martingale succeeds on it.

Schnorr proved that a sequence is Martin-Löf random if and only if no left-computable martingale succeeds on it. Therefore, one might be tempted to derive a similar relation between Schnorr random sequences and *computable* martingales. However, Schnorr pointed out that the increase in capital of a successful computable martingale can be so slow that it cannot be computably detected. Therefore, he introduced *order functions* ("Ordnungsfunktionen"), which ensure an effective control of the growth of the capital infinitely often.

In general, any nonnegative, real-valued, nondecreasing unbounded function g will be called an *order function*. (It should be remarked that, in Schnorr's terminology, an "Ordnungsfunktion" is always computable.)

Definition 9. Let $g : \mathbb{N} \to \mathbb{R}$ be a computable order function. A martingale is *g-successful* on a sequence $B \in 2^\omega$ if

$$d(B\restriction_n) \geq g(n) \quad \text{for infinitely many } n.$$

Schnorr showed that Schnorr null sets can be characterized via computable martingales successful against computable orders.

Theorem 4 (Schnorr)**.** *A set $\mathcal{X} \subseteq 2^\omega$ is Schnorr null if and only if there exists a computable martingale d and a computable order function g such that d is g-successful on all $B \in \mathcal{X}$.*

Observe that, in light of Theorem 2 (and the remark following (10)), a martingale being s-successful means it is g-successful for the order function $g(n) = 2^{(1-s)n}$. These are precisely what Schnorr calls *exponential orders*, so much of effective dimension is already, though apparently without explicit reference, present in Schnorr's treatment of algorithmic randomness [22].

Definition 10. For any sequence $B \in 2^\omega$, the *computable Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathrm{comp}} B$ and the *computable packing dimension* $\dim_{\mathrm{P}}^{\mathrm{comp}} B$ are defined as

$$\begin{aligned} \dim_{\mathrm{H}}^{\mathrm{comp}} B &= \inf\{s : \text{ some computable martingale } d \text{ is } s\text{-successful on } B\}. \\ \dim_{\mathrm{P}}^{\mathrm{comp}} B &= \inf\{s : \text{ some computable martingale } d \text{ is strongly } s\text{-successful on } B\}. \end{aligned}$$

Computable Hausdorff dimension was first explicitly defined in [12], computable packing dimension in [1].

A sequence is *computably random* if no computable martingale succeeds on it. [33] showed that the concepts of computable randomness and Schnorr randomness do not coincide. There are Schnorr random sequences on which some computable martingale succeeds. However, the differences vanish when it comes to dimension.

Theorem 5. *For any sequence $B \in 2^\omega$,*

$$\dim_{\mathrm{H}}^{\mathrm{S}} B = \dim_{\mathrm{H}}^{\mathrm{comp}} B$$

Proof. ($\leq$) Suppose a computable martingale d is s-successful on B. (We may assume that $s < 1$. The case $s = 1$ is trivial. We may also assume the d is normed.) It suffices to show that for any t such that $1 > t > s$ we can find a Schnorr t-test which covers B.

We define

$$U_k^{(t)} = \left\{ \sigma : \sigma \text{ is minimal such that } \frac{d(\sigma)}{2^{(1-t)|\sigma|}} \geq 2^k \right\}.$$

It is easy to see that the $(U_k^{(t)})_{k\in\mathbb{N}}$ cover B. Since d is computable, the cover is effective. To show that the measure of each $U_k^{(t)}$ is at most 2^{-k}, note that it easily follows inductively from the fairness property of martingales that for all prefix-free sets $V \subseteq 2^{<\omega}$,

$$\sum_{\sigma\in V} d(\sigma)2^{|\sigma|} \leq 1.$$

Therefore,

$$\sum_{\sigma\in U_k^{(t)}} 2^{(1-t)|\sigma|}2^k 2^{|\sigma|} \leq 1,$$

and hence

$$\sum_{\sigma\in U_k^{(t)}} 2^{-t|\sigma|} \leq 2^{-k}.$$

The only thing that is left to prove is that $\sum_{w\in U_k^{(t)}} 2^{-s|w|}$ is a computable real number.

To approximate $\sum_{w\in U_k^{(t)}} 2^{-s|w|}$ within 2^{-r}, effectively find a number n such that

$$2^{(1-t)n} \geq 2^r d(\epsilon).$$

If we enumerate only those strings σ into $U_k^{(t)}$ for which $|\sigma| \leq n$, we may conclude for the remaining strings $\tau \in U_k^{(t)}$ that $d(\tau) \geq 2^{(1-t)n}2^k \geq 2^{r+k}d(\epsilon)$.

We now employ an inequality for martingales, which is sometimes referred to as *Kolmogorov's inequality*, but was first shown by [32]. If d is a martingale, then it holds for every $k > 0$,

$$\lambda\{B \in 2^\omega : d(B\upharpoonright_n) \geq k \text{ for some } n\} \leq \frac{d(\epsilon)}{k},$$

where λ denotes Lebesgue measure on 2^ω. It follows that the measure induced by the strings not enumerated is at most $2^{-(r+k)}$.

($\geq$) Suppose $\dim_{\mathrm{H}}^{\mathrm{S}} B < s < 1$. (Again the case $s = 1$ is trivial.) We show that for any $t > s$, there exists a computable martingale d which is s-successful on B.

Let $(V_k)_{k\in\mathbb{N}}$ be a Schnorr t-test for B. Since each V_k is computable, we may assume each V_k is prefix-free. Let

$$d_k(\sigma) = \begin{cases} 2^{(1-s)|\upsilon|} & \text{if } \sigma \sqsupset \upsilon \text{ for some } \upsilon \in V_k, \\ \sum_{\sigma w\in V_k} 2^{-|w|+(1-s)(|\sigma|+|w|)} & \text{otherwise.} \end{cases}$$

We verify that d_k is a martingale. Given $\sigma \in 2^{<\omega}$, if there is a $\upsilon \in V_k$ such that $\upsilon \sqsubseteq \sigma$, we have

$$d_k(\sigma 0) + d_k(\sigma 1) = 2^{1+(1-s)|\upsilon|}.$$

If $\upsilon \sqsubset \sigma$, then $d_k(\sigma) = 2^{(1-s)|\upsilon|}$, hence $d_k(\sigma 0) + d_k(\sigma 1) = 2d_k(\sigma)$. If $\upsilon = \sigma$, then, by definition of d_k and the fact that V_k is prefix-free, $d_k(\sigma) = 2^{(1-s)|\sigma|}$, thus $d_k(\sigma 0) + d_k(\sigma 1) = 2d_k(\sigma)$ holds in this case, too.

If such v does not exist,

$$\begin{aligned} d_k(\sigma 0) + d_k(\sigma 1) &= \sum_{\sigma 0 w \in V_k} 2^{-|w|+(1-s)(|\sigma|+|w|+1)} + \sum_{\sigma 1 w \in V_k} 2^{-|w|+(1-s)(|\sigma|+|w|+1)} \\ &= \sum_{\sigma u \in V_k} 2^{(-|u|+1)+(1-s)(|\sigma|+|u|)} = 2 d_k(\sigma). \end{aligned}$$

Besides, $d_k(\epsilon) = \sum_{w \in V_k} 2^{-|w|+(1-s)|w|} = \sum_{w \in V_k} 2^{-s|w|} \leq 2^{-k}$, so the function

$$d = \sum_k d_k$$

defines a martingale as well (using additivity). Finally, note that, for $w \in V_k$, $d(w) \geq d_k(w) = 2^{(1-s)|w|}$. So if $B \in \bigcap_k [V_k]$, then $d(B\restriction_n) \geq 2^{(1-s)n}$ infinitely often, which means that d is s-successful on all $B \in \mathcal{X}$.

Since each $d_k(\epsilon) \leq 2^{-k}$, the computability of d follows easily from the uniform computability of each d_k, which is easily verified based on the fact that the measure of the V_k is uniformly computable. (Note that each σ can be in at most finitely many V_k.) □

An alternative proof of Theorem 5 could have been obtained by showing that a sequence B is Schnorr s-null if and only if there exists a computable order function g such that $d(B\restriction_n) \geq 2^{(1-s)n} g(n)$ infinitely often, i.e. by transferring Schnorr's characterization of Schnorr random sequences via computable martingales to the case of Hausdorff measures. Then Theorem 5 follows easily resorting to (10).

So, in contrast to randomness, the approach via Schnorr tests and the approach via computable martingales to dimension yield the same concept.

In the following, we will use both names, $\dim_{\mathrm{H}}^{\mathrm{S}}$ and $\dim_{\mathrm{H}}^{\mathrm{comp}}$, stressing whether the reasoning follows the test or martingale approach, respectively. Theorem 5 justifies to regard computable packing dimension as the dual to Schnorr Hausdorff dimension.

It follows from the definitions that for any sequence $A \in 2^{\omega}$, $\dim_{\mathrm{H}}^{\mathrm{comp}} A \leq \dim_{\mathrm{P}}^{\mathrm{comp}} A$. We call sequences for which computable Hausdorff and computable packing dimension coincide *computably regular*, following [31] and [1]. It is easy to construct a non-computably regular sequence, however, in Section 7 we will see that such sequences already occur among the class of c.e. sets.

5. Examples of Schnorr/Computable Dimension

The previous results allow to exhibit two typical examples of Schnorr dimension. They can be seen as 'pointwise' versions of Theorem 1 and are Schnorr dimension analogues for two canonical examples of sequences having nonintegral effective (constructive) dimension [1, 16]. The first example is obtained by 'inserting' zeroes into a sequence of dimension 1. Note that it easily follows from the definitions that every Schnorr random sequence has Schnorr Hausdorff dimension one. On the other hand, it is not hard to show that not every sequence of Schnorr Hausdorff dimension 1 is also Schnorr random.

The second class of examples is based on the fact that Schnorr random sequences satisfy the law of large numbers, not only with respect to Lebesgue measure (which corresponds to the uniform Bernoulli measure on 2^{ω}), but also with respect to other computable Bernoulli distributions. Given a sequence $\vec{p} = (p_n)_{n \in \mathbb{N}}$ of real numbers, where $p_n \in (0,1)$ for all n, the *Bernoulli measure* $\mu_{\vec{p}}$ is defined by setting

$$\mu_{\vec{p}}[\sigma] = \prod_{\sigma(i)=1} p_i \prod_{\sigma(i)=0} 1 - p_i .$$

The sequence $\vec{p}$ is called the *bias sequence* of $\mu_{\vec{p}}$. The measure $\mu_{\vec{p}}$ is *computable* if the bias sequence is a uniformly computable sequence of real numbers.

One can modify the definition of Schnorr tests to obtain randomness notions for arbitrary computable measures μ. Given a computable measure μ, a sequence is called Schnorr μ-random if it is not covered by any μ-Schnorr test.

Theorem 6. *(1) Let $S \in 2^\omega$ be Schnorr random, and let Z be a computable, infinite, co-infinite set of natural numbers such that $\delta_Z = \lim_n |\{0, \dots, n-1\} \cap Z|/n$ exists. Define a new sequence S_Z by*

$$S_Z \restriction_Z = S \quad \text{and} \quad S_Z \restriction_{\overline{Z}} = 0,$$

where 0 *here denotes the sequence consisting of zeroes only. Then it holds that*

$$\dim_{\mathrm{H}}^{\mathrm{S}} S_Z = \delta_Z.$$

(2) Let $\mu_{\vec{p}}$ be a computable Bernoulli measure on 2^ω with bias sequence $(p_0, p_1, \dots)$ such that $\lim_n p_n = p$. *Then it holds that for any Schnorr $\mu_{\vec{p}}$-random sequence B,*

$$\dim_{\mathrm{H}}^{\mathrm{S}} B = -[p \log p + (1-p)\log(1-p)].$$

Part (1) of the theorem is straightforward (using for instance the martingale characterization Theorem 5); part (2) is an easy adaption of the corresponding theorem for effective (i.e. Martin-Löf style) dimension (as for example in [16]).

It is not hard to see that for the examples given in Theorem 6, Hausdorff dimension and packing dimension coincide, so they describe regular sequences. In Section 7 we will see that there are highly irregular c.e. sets of natural numbers: While all c.e. sets have computable Hausdorff dimension 0, there are c.e. sets of computable packing dimension 1.

6. A MACHINE CHARACTERIZATION OF SCHNORR DIMENSION

One of the most cogent arguments in favor of Martin-Löf's approach to randomness is the coincidence of the Martin-Löf random sequences with the sequences that are incompressible in terms of prefix-free Kolmogorov complexity K. Furthermore, there exists a fundamental correspondence between effective Hausdorff and packing dimension, $\dim_{\mathrm{H}}^1$ and $\dim_{\mathrm{P}}^1$, respectively, and Kolmogorov complexity: For any sequence A it holds that

$$\dim_{\mathrm{H}}^1 A = \liminf_{n\to\infty} \frac{\mathrm{K}(A\restriction_n)}{n} \quad \text{and } \dim_{\mathrm{P}}^1 A = \limsup_{n\to\infty} \frac{\mathrm{K}(A\restriction_n)}{n}.$$

The first equation was first explicitly proved in [15], but much of it is already present in earlier works on Kolmogorov complexity and Hausdorff dimension, such as [19] or [25]. The second identity is due to [1].

Note that in both equations one could replace prefix-free complexity K by plain Kolmogorov complexity C, since both complexities differ only by a logarithmic factor.

To obtain a machine characterization of Schnorr dimension, we have to restrict the admissible machines to those with domains having computable measure. Recall that a Turing machine is *prefix-free* if its domain is.

Definition 11. A prefix-free machine M is *computable* if

$$\sum_{w\in\mathrm{dom}(M)} 2^{-|w|} \tag{12}$$

is a computable real number.

Note that, as in the case of Schnorr tests, if a machine is computable then its domain is computable (but not vice versa). To determine whether $M(w)\downarrow$, enumerate $\mathrm{dom}(M)$ until the value of $\sum_{w\in\mathrm{dom}(M)} 2^{-|w|}$ is approximated with precision 2^{-N}, where $N > |w|$. If $M(w)\downarrow$, then w must have been enumerated by this point.

The definition of machine complexity follows the standard scheme. We restrict ourselves to prefix-free machines.

Definition 12. Given a Turing machine M with prefix-free domain, the *M-complexity* of a string x is defined as

$$\mathrm{K}_M(x) = \min\{|p| : M(p) = x\},$$

where $\mathrm{K}_M(x) = \infty$ if there does not exist a $p \in 2^{<\omega}$ such that $M(p) = x$.

We refer to the books by [11] and [7] for comprehensive treatments on machine (Kolmogorov) complexity. Furthermore, following [6], we may assume that the measure of the domain of a computable machine is 1. Namely, for each computable prefix-free machine M there exists a prefix-free machine $\widetilde{M}$ such that $\lambda(\mathrm{dom}(\widetilde{M})) = 1$, and for all $\sigma \in \mathrm{range}(M)$, $\mathrm{K}_M(\sigma) = \mathrm{K}_{\widetilde{M}}(\sigma) + \mathrm{O}(1)$ (i.e. there exists a constant c such that for all $\sigma \in \mathrm{range}(M)$, $\mathrm{K}_M(\sigma)$ and $\mathrm{K}_{\widetilde{M}}(\sigma)$ differ by at most c). This can be justified by adding "superfluous" strings to the domain and applying the Kraft-Chaitin Theorem.

Our machine characterization of Schnorr dimension will be based on the following characterization of Schnorr randomness by [6].

Theorem 7 (Downey and Griffiths, 2004)**.** *A sequence A is Schnorr random if and only if for every computable machine M,*

$$(\exists c)\,(\forall n)\;\mathrm{K}_M(A\restriction_n) \geq n - c.$$

Building on this characterization, we can go on to describe Schnorr dimension as asymptotic entropy with respect to computable machines.

Theorem 8. *For any sequence A it holds that*

$$\dim_{\mathrm{H}}^{\mathrm{S}} A = \inf_M \underline{\mathrm{K}}_M(A) \quad \textit{where} \quad \underline{\mathrm{K}}_M(A) = \liminf_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n},$$

where the infimum is taken over all computable prefix-free machines M.

Proof. ($\geq$) Let $s > \dim_{\mathrm{H}}^{\mathrm{S}} A$. We show that this implies $s \geq \underline{\mathrm{K}}_M(A)$ for some computable machine M, which yields $\dim_{\mathrm{H}}^{\mathrm{S}} A \geq \inf_M \underline{\mathrm{K}}_M(A)$.

As $s > \dim_{\mathrm{H}}^{\mathrm{S}} A$, there exists a Schnorr s-test $\{U_i\}$ such that $A \in \bigcap_i [U_i]$. Assume each set in the test is given as $U_n = \{\sigma_{n,1}, \sigma_{n,2}, \dots\}$. Note that the Kraft-Chaitin Theorem is applicable to the set of axioms

$$\langle \lceil s|\sigma_{n,i}|\rceil - 1, \sigma_{n,i}\rangle \quad (n \geq 2,\; i \geq 1).$$

Hence there exists a prefix-free machine M such that for $n \geq 2$ and all i, $\mathrm{K}_M(\sigma_{n,i}) = \lceil s|\sigma_{n,i}|\rceil - 1$. Furthermore, M is computable since $\sum_{n,i} 2^{-\lceil s|\sigma_{n,i}|\rceil - 1}$ is computable.

We know that for all n there is an i_n such that $\sigma_{n,i_n} \sqsubset A$, and it is easy to see that the length of these σ_{n,i_n} goes to infinity. Hence there must be infinitely many $m = |\sigma_{n,i_n}|$ such that

$$\mathrm{K}_M(A\restriction_m) \leq \lceil s|\sigma_{n,i_n}|\rceil - 1 \leq sm,$$

which in turn implies that

$$\liminf_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n} \leq s.$$

($\leq$) Suppose $s > \inf_M \underline{\mathrm{K}}_M(A)$. So there exists a computable prefix-free machine M such that $s > \underline{\mathrm{K}}_M(A)$. Define the set

$$S_M = \{w \in 2^{<\omega} : \ \mathrm{K}_M(w) < |w|s\}.$$

We claim that this is a total Solovay s-cover for A. It is obvious that the set covers A infinitely often, so it remains to show that

$$\sum_{w \in S_M} 2^{-|w|s}$$

is a computable real number less than or equal to 1. The latter follows from

$$\sum_{w \in S_M} 2^{-|w|s} < \sum_{w \in S_M} 2^{-K_M(w)} \leq 1,$$

by Kraft's inequality and the fact that M is a prefix-free machine. To show computability, given ε compute the measure induced by $\mathrm{dom}(M)$ up to precision ε, so all strings not enumerated by that stage (call it t) will add in total at most ε to the measure of $\mathrm{dom}(M)$, which means they will also add at most ε to $\sum_{w \in S_M} 2^{-|w|s}$, hence

$$\sum_{w \in S_{M_t}} 2^{-|w|s} \leq \sum_{w \in S_M} 2^{-|w|s} \leq \sum_{w \in S_{M_t}} 2^{-|w|s} + \varepsilon,$$

since a v contributes to S_M only if $\mathrm{K}(v) < |v|s$. But obviously, this only happens if $v \in \mathrm{dom}(M)$. □

A similar machine characterization of Schnorr Hausdorff dimension was independently obtained by [10]. One can use a similar argument to obtain a machine characterization of computable packing dimension.

Theorem 9. *For any sequence A it holds that*

$$\dim_{\mathrm{P}}^{\mathrm{comp}} A = \inf_M \overline{\mathrm{K}}_M(A) \quad \textit{where} \quad \overline{\mathrm{K}}_M(A) = \limsup_{n \to \infty} \frac{\mathrm{K}_M(A\restriction_n)}{n},$$

where the infimum is taken over all computable prefix-free machines M.

Proof. ($\geq$) Let $s > \dim_{\mathrm{P}}^{\mathrm{comp}} A$. We show that this implies $s \geq \overline{\mathrm{K}}_M(A)$ for some computable machine M, which yields $\dim_{\mathrm{P}}^{\mathrm{comp}} A \geq \inf_M \overline{\mathrm{K}}_M(A)$.

So assume d is a computable martingale that is strongly t-successful on A for some $t < s$. For each n, consider the set

$$U_n = \{w \in \{0,1\}^n : \ d(w) \geq 2^{(1-t)n}\}.$$

Then A is covered by all but finitely many U_n. Furthermore, the U_n are uniformly computable, as is the measure of each $[U_n]$. It follows from Kolmogorov's inequality that $|U_n| \leq 2^{nt}$. Hence

$$\sum_{w \in U_n} 2^{-|w|s} \leq 2^{n(t-s)}.$$

Since $t - s < 0$, we can choose an n_0 such that $\sum_{n \geq n_0} \sum_{w \in U_n} 2^{-|w|s} \leq 1/2$. Let $U = \bigcup_{n \geq n_0} U_n$. We can build a Kraft-Chaitin set based on the axioms

$$\langle \lceil s|w| \rceil - 1, w \rangle, \quad w \in U.$$

Then there exists a prefix-free machine M such that for all $w \in U$, $\mathrm{K}_M(w) \le s|w|$. Furthermore, M is computable since $\sum_{w \in U} 2^{-|w|s}$ is computable. But for $n \ge n_0$, every prefix $A\restriction_n$ is in U, and hence

$$\limsup_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n} \le s.$$

($\le$) Suppose $s > \inf_M \overline{\mathrm{K}}_M(A)$. So there exists a computable prefix-free machine M such that $s > \overline{\mathrm{K}}_M(A)$. For each n, define the set

$$U_n = \{w \in \{0,1\}^n : \mathrm{K}_M(w) < |w|s\}.$$

Again, A is covered by all but finitely many U_n^M. For each n, define a martingale d_n as in the $\ge$-part of the proof of Theorem 5. The d_n are uniformly computable as the U_n are. We use a fundamental result by [5]: For any $n, k \in \mathbb{N}$,

$$|\{w \in \{0,1\}^n : \mathrm{K}(w) \le k\}| \le 2^{k - \mathrm{K}(n) + \mathrm{O}(1)}.$$

Since M is prefix-free, $\mathrm{K} \le \mathrm{K}_M + \mathrm{O}(1)$, and hence the above inequality holds with K_M in place of K, too.

It follows that for some constant c and each n,

$$d_n(\epsilon) = \sum_{w \in U_n} 2^{-s|w|} = |U_n| 2^{-sn} \le 2^{-\mathrm{K}(n)+c}.$$

So $d = \sum_n d_n$ is well-defined, since $\sum_n 2^{-\mathrm{K}(n)}$ is finite. A is covered by all but finitely many U_n, and for $w \in U_n$, $d(w) \ge d_n(w) = 2^{(1-s)|w|}$, so d is strongly s-successful on A. □

7. Schnorr Dimension and Computable Enumerability

Usually, when studying algorithmic randomness, interest focuses on *left-computable real numbers* (also known as *c.e. reals*) rather than on *c.e. sets* (of natural numbers). The reason is that c.e. sets exhibit a trivial behavior with respect to most randomness notions, while there are c.e. reals which are random, such as Chaitin's Ω.

As regards left-computable reals, with respect to computability, so far all notions of effective dimension show mostly the same behavior as the corresponding notions of randomness. For instance, it has been shown by [16] and [30] that every left-computable real of positive effective dimension is Turing-complete, a result that was known before to hold for left-computable Martin-Löf random reals. For Schnorr dimension, a straightforward generalization of a proof by [6], who showed that every left-computable Schnorr random real is of high degree, yields that the same holds true for left-computable reals of positive Schnorr Hausdorff dimension. That is, if A is left-computable and $\dim_{\mathrm{H}}^{\mathrm{S}} A > 0$, then $A' \equiv_{\mathrm{T}} 0''$.

As regards *computably enumerable sets* (of natural numbers), they are usually, in the context of algorithmic randomness, of marginal interest, since they exhibit a rather non-random behavior. For instance, it is easy to see that no computably enumerable set can be Schnorr random.

Proposition 3. *No computably enumerable set is Schnorr random.*

Proof. Every infinite c.e. set contains an infinite computable subset. So, given an infinite c.e. set $A \subseteq \mathbb{N}$, choose some computable infinite subset B. Assume $B = \{b_1, b_2, \dots\}$, with $b_i < b_{i+1}$.

Define a Schnorr test $\{V_n\}$ for A as follows: At level n, put all those strings υ of length $b_n + 1$ into V_n for which

$$\upsilon(b_i) = 1 \quad \text{for all } i \leq n+1.$$

Then surely $A \in [V_n]$ for all n, and $\lambda[V_n] = 2^{-n}$. □

It does not seem clear how to improve the preceding result to Schnorr dimension zero. Indeed, defining coverings from the enumeration of a set directly might not work, because due to the dimension factor in Hausdorff measures, longer strings will be weighted higher. Depending on how the enumeration is distributed, this might not lead to a Schnorr s-covering at all.

However, one can exploit the somewhat predictable nature of a c.e. set to define a computable martingale which is, for any $s > 0$, s-successful on the characteristic sequence of the enumerable set, thereby ensuring that each c.e. set has computable dimension 0.

Theorem 10. *Every computably enumerable set $A \subseteq \mathbb{N}$ has Schnorr Hausdorff dimension zero.*

Proof. Given rational $s > 0$, we show that there exists a computable martingale d such that d is s-successful on A.

First, partition the natural numbers into effectively given, disjoint intervals I_n such that $|I_n| \ll |I_{n+1}|$, for instance, $|I_n| = 2^{|I_0|+\cdots+|I_{n-1}|}$. Set $i_n = |I_n|$ and $j_n = i_0 + i_1 + \cdots i_n$.

Denote by δ the upper density of A on I_n, i.e.

$$\delta = \limsup_{n\to\infty} \frac{|A \cap I_n|}{i_n}.$$

W.l.o.g. we may assume that $\delta > 0$. For any $\varepsilon > 0$ with $\varepsilon < \delta$ there is a rational number r such that $\delta - \varepsilon < r < \delta$. Given such an r, there must be infinitely many n_k for which

$$|A \cap I_{n_k}| > r i_{n_k}.$$

Define a computable martingale d by describing an accordant betting strategy as follows. At stage 0, initialize with $d(\epsilon) = 1$. At stage $k+1$, assume d is defined for all τ with $|\tau| \leq l_k$ for some $l_k \in \mathbb{N}$. Enumerate A until we know that for some interval I_{n_k} with $j_{n_k-1} > l_k$ (i.e. I_{n_k} has not been bet on before),

$$|A \cap I_{n_k}| > r i_{n_k}.$$

For all strings σ with $l_k < |\sigma| \leq j_{n_k-1}$, bet nothing (i.e. d remains constant here). Fix a (rational) stake $\gamma > 2^{1-s} - 1$. On I_{n_k}, bet γ on the mth bit being 1 ($j_{n_k-1} < m \leq j_{n_k}$) if m has already been enumerated into A. Otherwise bet γ on the mth bit being 0. Set $l_{k+1} = j_{n_k}$.

When betting against A, obviously this strategy will lose at most $\lceil 2\varepsilon \rceil |I_{n_k}|$ times on I_{n_k}. Thus, for all sufficiently large n_k,

$$\begin{aligned} d(A\restriction_{l_{k+1}}) &\geq d(A\restriction_{l_k})(1+\gamma)^{i_{n_k} - \lceil 2\varepsilon \rceil i_{n_k}} (1-\gamma)^{\lceil 2\varepsilon \rceil i_{n_k}} \\ &= d(A\restriction_{l_k})(1+\gamma)^{i_{n_k}} \left(\frac{1-\gamma}{1+\gamma}\right)^{\lceil 2\varepsilon \rceil i_{n_k}} > 2^{(1-s) i_{n_k}} \left(\frac{1-\gamma}{1+\gamma}\right)^{\lceil 2\varepsilon \rceil i_{n_k}}. \end{aligned}$$

Choosing ε small and n large enough we see that d is s-successful on A. □

On the other hand, it is not hard to see that for every Schnorr 1-test there is a c.e. set which is not covered by it. This means that the class of all c.e. sets has Schnorr Hausdorff dimension 1. For effective Hausdorff dimension, [14] showed that for any class $\mathcal{X} \subseteq 2^\omega$,

$$\dim_{\mathrm{H}}^1 \mathcal{X} = \sup\{\dim_{\mathrm{H}}^1 A : A \in \mathcal{X}\}.$$

This means that effective dimension has a strong *stability* property. The class of c.e. sets yields an example where stability fails for Schnorr dimension.

In contrast to Theorem 10, perhaps somewhat surprisingly, the upper Schnorr entropy of c.e. sets can be as high as possible, namely, there exist c.e. sets with computable packing dimension 1. This stands in sharp contrast to the case of effective dimension, where Barzdiņš' Theorem [1968] ensures that all c.e. sets have effective packing dimension 0. Namely, it holds that if A is a c.e. set, then there exists a c such that for all n, $\mathrm{C}(A\restriction_n) \leq \log n + c$.

In fact, it can be shown that every hyperimmune degree contains a set of computable packing dimension 1. As the proof of the theorem shows, this holds mainly because of the requirement that all machines involved in the determination of Schnorr dimension are total.

Before giving the proof, however, it should be mentioned that there are degrees which do not contain any sequence of high computable packing dimension. This can be shown by a straightforward construction.

Theorem 11. *For any hyperimmune set B there exists a set $A \equiv_{\mathrm{T}} B$ such that*

$$\dim_{\mathrm{P}}^{\mathrm{comp}} A = 1.$$

Furthermore, if the set B is c.e., then A can be chosen to be c.e., too.

Proof. For given B, it suffices to construct a set $C \leq_{\mathrm{T}} B$ such that $\dim_{\mathrm{P}}^{\mathrm{comp}} C = 1$ and to let, for some computable set of places Z of sublinear density, the set A be a join of B and C where B is coded into the places in Z in the sense that

$$A\restriction_Z = B \quad \text{and} \quad A\restriction_{\overline{Z}} = C;$$

a similar argument works for the case of c.e. sets.

So fix any hyperimmune set B. Then there is a function g computable in B such that for any computable function f there are infinitely many n such that $f(n) < g(n)$. Partition the natural numbers into effectively given, pairwise disjoint intervals

$$\mathbb{N} = I_0 \cup I_1 \cup I_2 \cup \ldots$$

such that $|I_0| + \ldots + |I_n| \ll |I_{n+1}|$ for all n; for instance, choose I_n such that $|I_{n+1}| = 2^{|I_0|+\cdots+|I_n|}$, and let $i_n = |I_n|$. Furthermore, let $M_0, M_1, \ldots$ be a standard enumeration of all prefix-free (not necessarily computable) Turing machines with uniformly computable approximations $M_e[s]$.

For any pair of indices e and n, let C have an empty intersection with the interval $I_{\langle e,n\rangle}$ in case

$$\sum_{M_e[g(n)](w)\downarrow} 2^{-|w|} \leq 1 - 2^{-i_{\langle e,n\rangle}}. \tag{13}$$

Otherwise, in case (13) is false, any string of length $i_{\langle e,n\rangle}$ not output by M_e at stage $g(n)$ via an M_e-program of length at most $i_{\langle e,n\rangle}$ is M_e-incompressible in the sense that the string has M_e-complexity of at least $i_{\langle e,n\rangle}$; pick such a string σ and let $C\restriction_{I_{\langle e,n\rangle}} = \sigma$ (in case such a string does not exist, the domain of the prefix-free machine M_e contains exactly the finitely many strings of length $i_{\langle e,n\rangle}$ and we don't have to worry about M_e). Observe that $C \leq_{\mathrm{T}} B$ because g is computable in B.

For any M_e with domain of measure one, the function f_e that maps n to the first stage t such that

$$\sum_{M_e[t](w)\downarrow} 2^{-|w|} > 1 - 2^{-i_{\langle e,n\rangle}} \tag{14}$$

is total and in fact computable; hence there are infinitely many n such that $f_e(n) < g(n)$ and for all these n, the restriction of C to $I_{\langle e,n\rangle}$ is M_e-incompressible. To see that this ensures computable packing dimension 1, suppose

$$\dim_{\mathrm{P}}^{\mathrm{comp}} C < 1.$$

Then there exists a computable machine M, an $\varepsilon > 0$ and some $n_\varepsilon \in \mathbb{N}$ such that

$$(\forall n \geq n_\varepsilon)\,[\mathrm{K}_M(C\restriction_n) \leq (1-\varepsilon)n].$$

We define another total machine $\widetilde{M}$ with the same domain as M: Given x compute $M(x)$. If $M(x)\downarrow$, check whether $|M(x)| = i_0 + i_1 + \cdots + i_k$ for some k. If so, output the last i_k bits, otherwise output 0. Let e be an index of $\widetilde{M}$. By choice of the i_k, for all sufficiently large n, the $\widetilde{M}$-complexity of $C\restriction_{I_{\langle e,n\rangle}}$ can be bounded as follows

$$\mathrm{K}_{\widetilde{M}}(C\restriction_{I_{\langle e,n\rangle}}) \leq \mathrm{K}_M(C\restriction_{I_{\langle e,0\rangle}\cup\ldots\cup I_{\langle e,n\rangle}}) \leq (1-\varepsilon)(i_{\langle e,0\rangle} + \cdots + i_{\langle e,n\rangle}), \leq (1-\frac{\varepsilon}{2})i_{\langle e,n\rangle}$$

which contradicts the fact that by construction there are infinitely many n such that the restriction of C to the interval $I_{\langle e,n\rangle}$ is M_e-incompressible, that is, $\widetilde{M}$-incompressible.

In the case of a noncomputable c.e. set B, it is not hard to see that we obtain a function g as above if we let $g(n)$ be equal to the least stage such that some fixed effective approximation to B agrees with B at place n. Using this function g in the construction above, the set C becomes c.e. because for any index e and for all n, in case n is not in B the restriction of C to the interval $I_{\langle e,n\rangle}$ is empty, whereas otherwise it suffices to wait for the stage $g(n)$ such that n enters B and to compute from $g(n)$ the restriction of C to the interval $I_{\langle e,n\rangle}$, then enumerating all the elements of C in this interval. □

8. Acknowledgments

We would like to thank John Hitchcock for some helpful remarks. We would also like to thank an anonymous referee for many detailed and helpful comments.

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