

The Completeness Theorem

Key Concepts

- The **Completeness Theorem** states that

$$T \vdash_{\mathcal{L}} \varphi \iff T \vDash \varphi$$

- The \Rightarrow -direction is **Soundness**, which we already proved.
- The \Leftarrow -direction is usually proved in the following form:

If T is consistent, then T has a model.

- To construct a model of a consistent theory, we consider the **constant terms**

$$K := \{t : t \text{ } \mathcal{L}\text{-term without variables}\}$$

and **identify provably equal terms**:

$$s \sim t : \iff T \vdash_{\mathcal{L}} s = t$$

- The **canonical term structure** \mathcal{A} of T has universe $A = K / \sim$.
- Moreover, we put

- $c^{\mathcal{A}} := [c]$
- $f^{\mathcal{A}}([t_1], \dots, [t_n]) := [ft_1 \dots t_n]$
- $R^{\mathcal{A}}([t_1], \dots, [t_n]) : \iff T \vdash_{\mathcal{L}} Rt_1 \dots t_n$

- It holds that for any *atomic* sentence σ ,

$$\mathcal{A} \vDash \sigma \iff T \vdash \sigma$$

Problems

Discuss

We all believe (I think) that $(\mathbb{Z}, +, 0)$ is a model of the group axioms. To what extent does this prove that the group axioms are consistent?

Exercise 0.1.

Verify that \sim is an equivalence relation.

Exercise 0.2.

Verify that the definition of $c^{\mathcal{A}}, f^{\mathcal{A}}, R^{\mathcal{A}}$ does not depend on the choice of representative for $[c]$ and $[t_i]$.

Exercise 0.3.

Verify the claim that for atomic sentences,

$$\mathcal{A}_T \vDash \sigma \iff T \vdash \sigma$$