

The Diagonal Lemma

In the following, we consider the **diagonal function** d , defined by

$$d(n) = \begin{cases} \ulcorner \forall y (y = \underline{n} \rightarrow \sigma(y)) \urcorner & \text{if } n = \ulcorner \sigma(v_0) \urcorner \text{ for an } L\text{-formula } \sigma(v_0) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 0.1.

Let T be an L -theory, $\theta(v_0)$ an L -formula with the single free variable v_0 . If the diagonal function d is representable in T , then there exists an L -sentence G with the property

$$T \vdash G \leftrightarrow \theta(\ulcorner G \urcorner).$$

If $\theta(v_0)$ is a Π_1 -formula, then G can also be chosen as a Π_1 -sentence.

Proof. If d is represented in T by the formula $\delta(x, y)$, then define

$$\psi(v_0) := \forall y (\delta(v_0, y) \rightarrow \theta(y)),$$

and set $n := \ulcorner \psi(v_0) \urcorner$.

Note: n is actually a number in which the variable v_0 does not occur; however, the Gödel number of v_0 does enter into the calculation of n .

For G , we now choose the sentence

$$G := \forall y (y = \underline{n} \rightarrow \psi(y)).$$

Then G has Gödel number $d(\ulcorner \psi(v_0) \urcorner)$. If $\delta(x, y)$ is a Σ_1 -formula and $\theta(v)$ is a Π_1 -formula, then ψ and G are (equivalent to) Π_1 -formulas. Thus it remains to show that $T \vdash G \leftrightarrow \theta(\ulcorner G \urcorner)$:

It is clear that

$$\begin{aligned} T \vdash G &\leftrightarrow \psi(\underline{n}), && \text{i.e., by the definition of } \psi \\ (1) \quad T \vdash G &\leftrightarrow \forall y (\delta(\underline{n}, y) \rightarrow \theta(y)). \end{aligned}$$

But the fact that d is represented in T by $\delta(x, y)$ means

$$d(n) = \ulcorner G \urcorner \Rightarrow T \vdash \delta(\underline{n}, \ulcorner G \urcorner) \quad \text{and} \quad T \vdash \exists! y \delta(\underline{n}, y).$$

Thus in particular

(2) $T \vdash \forall y (\delta(\underline{n}, y) \leftrightarrow y = \ulcorner G \urcorner)$ and together with (1) it follows
 $T \vdash G \leftrightarrow \forall y (y = \ulcorner G \urcorner \rightarrow \theta(y))$, hence
 $T \vdash G \leftrightarrow \theta(\ulcorner G \urcorner)$.

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